

# Convex Hull of Planar H-Polyhedra

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## Abstract

Suppose  $\langle A_i, \vec{c}_i \rangle$  are planar (convex) H-polyhedra, that is,  $A_i \in \mathbb{R}^{n_i \times 2}$  and  $\vec{c}_i \in \mathbb{R}^{n_i}$ . Let  $P_i = \{\vec{x} \in \mathbb{R}^2 \mid A_i \vec{x} \leq \vec{c}_i\}$  and  $n = n_1 + n_2$ . We present an  $O(n \log n)$  algorithm for calculating an H-polyhedron  $\langle A, \vec{c} \rangle$  with the smallest  $P = \{\vec{x} \in \mathbb{R}^2 \mid A \vec{x} \leq \vec{c}\}$  such that  $P_1 \cup P_2 \subseteq P$ .

**Keywords:** convex hull, computational geometry

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## 1 Introduction

The convex hull problem is classically stated as the problem of computing the minimal convex region that contains  $n$  distinct points  $\{(x_i, y_i)\}_{i=1}^n$  in the Euclidean plane  $\mathbb{R}^2$ . The seminal work of Graham [4] showed that the convex hull problem can be solved in  $O(n \log n)$  worst-case running time. It inspired many to elaborate on, for example, the three and more dimensional case, specialised algorithms for polygons, on-line variants, *etc.* [8, 10]. The convex hull of polytopes (bounded polyhedra) can be calculated straightforwardly by taking the convex hull of their extreme points. However, calculating the convex hull for polyhedra turns out to be more subtle due to a large number of geometric configurations. Even for planar polyhedra, the introduction of rays makes it necessary to handle polyhedra such as a single half-space, a single ray, a single line, two facing (not coinciding) half-spaces, *etc.*, all of which require special handling in a point-based algorithm. The problem is exacerbated by the number of ways these special polyhedra can be combined. In contrast, we present a direct reduction of the convex hull problem of planar polyhedra to the convex hull problem for a set of points [4]. By confining all input points to a box

and applying the rays to translate these points outside the box, a linear pass around the convex hull of all these points is sufficient to determine the resulting polyhedron. By adopting the classic Graham scan algorithm, our algorithm inherits its  $O(n \log n)$  time complexity. The standard tactic for calculating the convex hull of  $H$ -polyhedra is to convert the input into an intermediate ray and vertex representation. Two common approaches to this conversion problem are the double description method [7] (also known as the Chernikova algorithm [3]) and the vertex enumeration algorithm of Avis and Fukuda [2]. The Chernikova method leads to a cubic time solution for calculating the convex hull of planar  $H$ -polyhedra [6] whereas the Avis and Fukuda approach runs in quadratic time.

The remaining sections are organised as follows: A self-contained overview of the algorithm, together with a worked example, is given in the next Section. A formal proof of correctness is given in Section 3. Section 4 concludes.

## 2 Planar Convex Hull Algorithm

The planar convex hull algorithm takes as input two  $H$ -polyhedra and outputs the smallest  $H$ -polyhedron which includes the input. The  $H$ -representation of a planar polyhedron corresponds to a set of inequalities each of which takes the form  $ax + by \leq c$ , where  $a, b, c \in \mathbb{R}$  and either  $a \neq 0$  or  $b \neq 0$ . Let  $Lin$  denote the set of all such inequalities. The vector  $\langle a, b \rangle$  is orthogonal to the boundary of the halfspace induced by  $ax + by \leq c$  and points away from the feasible space. This vector induces an ordering on halfspaces via the orientation mapping  $\theta$ . This map  $\theta : Lin \rightarrow [0, 2\pi)$  is defined such that  $\theta(ax + by \leq c) = \psi$  where  $\cos(\psi) = a/\sqrt{a^2 + b^2}$  and  $\sin(\psi) = b/\sqrt{a^2 + b^2}$ . The mapping  $\theta$  corresponds to the counter-clockwise angle which the half-space of  $x \leq 0$  has to be turned through to coincide with that of  $ax + by \leq c$ . Sorting half-spaces by angle is the key to efficiency in our algorithm. However,  $\theta$  is only used for comparing the orientation of two half-spaces. To aid the explanation of the algorithm, the concept of angular difference  $e_1 \angle e_2$  between two inequalities  $e_1$  and  $e_2$  is introduced as the counter-clockwise angle between  $\theta(e_1)$  and  $\theta(e_2)$ . More precisely  $e_1 \angle e_2 = (\theta(e_2) - \theta(e_1)) \bmod 2\pi$ . Note that this comparator can be realized without recourse to trigonometric functions [9].

The algorithm makes use of a number of simple auxiliary functions. The function  $intersect(a_1x + b_1y \leq c_1, a_2x + b_2y \leq c_2)$  calculates the set of intersection points of the two lines  $a_1x + b_1y = c_1$  and  $a_2x + b_2y = c_2$ . In practice an implementation of this function only needs to be partial since it is only applied in the algorithm when the result set contains a single point. The remaining auxiliaries are listed in Figure 1. The  $connect$  function generates an inequality from two points subject to the following constraints: the halfspace induced by  $connect(p_1, p_2)$  has  $p_1$  and  $p_2$  on its boundary and if  $p_1, p_2, p_3$  are ordered counter-clockwise then  $p_3$  is in the feasible space. The notation  $\overline{p_1, p_2}$  is used to abbreviate  $connect(p_1, p_2)$ . Furthermore, the predicate  $saturates(p, e)$  holds whenever the point  $p$  is on the boundary of the halfspace defined by the inequality  $e$ . Finally, the predicate  $inBox(s, p)$  determines whether the point  $p$  occurs

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1 function extreme( $E$ ) begin
2   sort  $E$  to obtain  $e_0 \dots, e_{n-1}$  such that  $\theta(e_0) < \theta(e_1) < \dots < \theta(e_{n-1})$ ;
3    $V := R := \emptyset$ ;
4   if  $E = \{ax + by \leq c\}$  then  $R := \{-a/\sqrt{a^2 + b^2}, -b/\sqrt{a^2 + b^2}\}$ ;
5   for  $i \in [0, n - 1]$  do let  $e_i \equiv ax + by \leq c$  in begin
6     // are the intersection points of this inequality degenerated?
7      $d_{pre} := (\theta(e_i) - \theta(e_{i-1 \bmod n})) \bmod 2\pi \geq \pi \vee n = 1$ ;
8      $d_{post} := (\theta(e_{i+1 \bmod n}) - \theta(e_i)) \bmod 2\pi \geq \pi \vee n = 1$ ;
9     if  $d_{pre}$  then  $R := R \cup \{b/\sqrt{a^2 + b^2}, -a/\sqrt{a^2 + b^2}\}$ ;
10    if  $d_{post}$  then  $R := R \cup \{-b/\sqrt{a^2 + b^2}, a/\sqrt{a^2 + b^2}\}$ ;
11    else  $V := V \cup \text{intersect}(e_i, e_{(i+1) \bmod n})$ ;
12    if  $d_{pre} \wedge d_{post}$  then  $V := V \cup \{v\}$  where  $v \in \{x, y \mid ax + by = c\}$ 
13  end
14  return  $\langle V, R \rangle$ 
15 end
16
17 function connect( $\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle$ )
18   return  $(y_2 - y_1)x + (x_1 - x_2)y \leq (y_2 - y_1)x_1 + (x_1 - x_2)y_1$ 
19
20 function saturates( $\langle x_1, y_1 \rangle, ax + by \leq c$ )
21   return  $(ax_1 + by_1 = c)$ 
22
23 function inBox( $s, \langle x, y \rangle$ )
24   return  $|x| < s \wedge |y| < s$ 

```

Figure 1: Convex hull algorithm for planar polyhedra

within a square of width  $2s$  that is centred on the origin.

The algorithm divides into a decomposition and a reconstruction phase. The *hull* function decomposes the input polyhedra into their corresponding ray and vertex representations by calling the function *extreme* in lines 3 and 4. The remainder of the *hull* function reconstructs a set of inequalities whose halfspaces enclose both sets of rays and points. The functions *extreme* and *hull* are presented in Figures 1 and 2, respectively. The algorithm requires the input polyhedra to be non-redundant. This means that no proper subset of the inequalities induces the same space as the original set of inequalities. The algorithm itself produces a non-redundant system.

To illustrate the algorithm consider Figure 3. The polyhedron  $E = \{e_0, e_1, e_2\}$  and the polytope  $E' = \{e'_0, \dots, e'_5\}$  constitute the input to the *hull* function. They are passed to the function *extreme* at line 28 and 29. Within *extreme* the inequalities of each polyhedron are sorted at line 2. Note that for ease of presentation the indices coincide with the angular ordering. The loop at lines 5–13 examines the relationship of each inequality with its two angular neighbours. If  $d_{post}$  is false, the intersection point  $\text{intersect}(e_i, e_{(i+1) \bmod n})$  is a vertex which is

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25 function hull( $E_1, E_2$ ) begin
26   // assertion: each  $E_i$  is satisfiable and non-redundant
27   if  $E_1 = \emptyset \vee E_2 = \emptyset$  then return  $\emptyset$ ;
28    $\langle P_1, R_1 \rangle := \textit{extreme}(E_1)$ ;
29    $\langle P_2, R_2 \rangle := \textit{extreme}(E_2)$ ;
30    $P := P_1 \cup P_2$ ;
31    $R := R_1 \cup R_2$ ; // Note:  $|R| \leq 8$ 
32    $s := \max\{|x|, |y| \mid \langle x, y \rangle \in P\} + 1$ ;
33   // add a point along the ray, that goes through  $x, y$ 
34   // and is outside the box
35    $Q := P$ ;
36   for  $\langle x, y, a, b \rangle \in P \times R$  do  $Q := Q \cup \{\langle x + 2\sqrt{2}sa, y + 2\sqrt{2}sb \rangle\}$ ;
37   // construct four inequalities in the zero dimensional case
38   if  $Q = \{\langle x_1, y_1 \rangle\}$  then return  $\{x \leq x_1, y \leq y_1, -x \leq -x_1, -y \leq -y_1\}$ ;
39   // the centre of gravity  $q_p$  is feasible but not a vertex (since  $|Q| > 1$ )
40    $q_p := \langle \sum_{\langle x, y \rangle \in Q} x / |Q|, \sum_{\langle x, y \rangle \in Q} y / |Q| \rangle$ ;
41   //  $q_p$  is pivot point for sorting:  $\forall i \in [0, n-2] . \theta(\overline{q_p, q_i}) \leq \theta(\overline{q_p, q_{i+1}})$ 
42    $\langle q_0, \dots, q_{n-1} \rangle := \textit{sort}(q_p, Q)$ 
43   // identify the  $m$  vertices  $q_{k_i}$  where  $0 \leq k_0 < \dots < k_{m-1} < n$ 
44    $\langle q_{k_0}, \dots, q_{k_{m-1}} \rangle := \textit{scan}(\langle q_0, \dots, q_{n-1} \rangle)$ 
45    $E_{\text{res}} := \emptyset$ ;
46   for  $i \in [0, m-1]$  do begin
47     let  $\langle x_1, y_1 \rangle = q_{k_i}, \langle x_2, y_2 \rangle = q_{k_{(i+1) \bmod m}}$ 
48     let  $e = \textit{connect}(\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle)$ 
49     // add  $e$  to  $E_{\text{res}}$  if  $q_{k_i}$  or  $q_{k_{(i+1) \bmod m}}$  is in the box...
50      $\textit{add} := \textit{inBox}(s, \langle x_1, y_1 \rangle) \vee \textit{inBox}(s, \langle x_2, y_2 \rangle) \vee m = 2$ ;
51      $j := (k_i + 1) \bmod n$ ;
52     while  $\neg \textit{add} \wedge j \neq k_{i+1}$  do begin
53       // ...or any boundary point is in the box
54        $\textit{add} := \textit{saturates}(q_j, e) \wedge \textit{inBox}(s, q_j)$ ;
55        $j := (j + 1) \bmod n$ ;
56     end;
57     if  $m = 2 \wedge \textit{inBox}(s, \langle x_1, y_1 \rangle)$  then
58       if  $y_1 = y_2$  then  $E_{\text{res}} := E_{\text{res}} \cup \{\textit{sgn}(x_1 - x_2)x \leq \textit{sgn}(x_1 - x_2)x_1\}$ 
59       else  $E_{\text{res}} := E_{\text{res}} \cup \{\textit{sgn}(y_1 - y_2)y \leq \textit{sgn}(y_1 - y_2)y_1\}$ 
60     if  $\textit{add}$  then  $E_{\text{res}} := E_{\text{res}} \cup \{e\}$ ;
61     end
62   end
63   return  $E_{\text{res}}$ 
64 end

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Figure 2: Convex hull algorithm for planar polyhedra

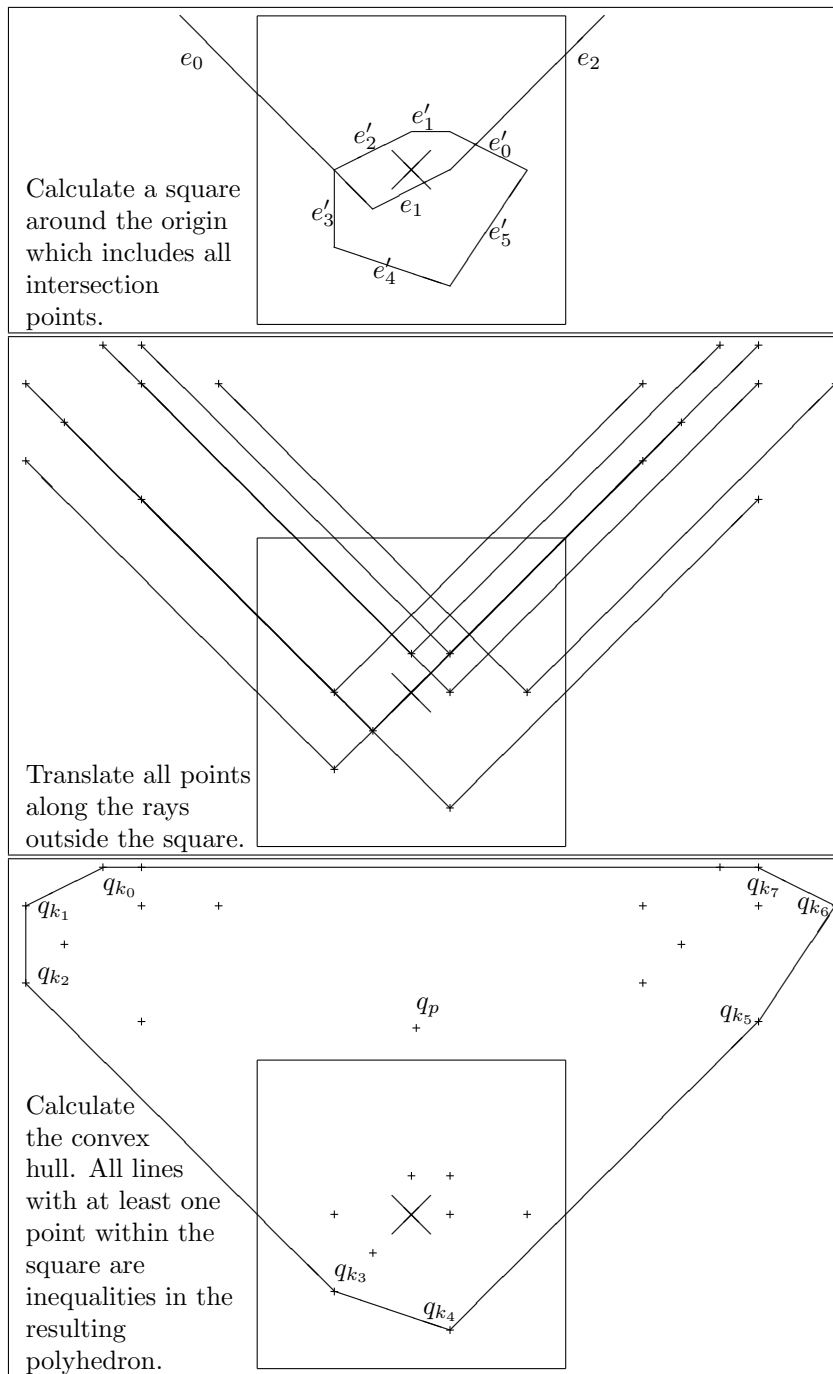


Figure 3: The different stages of the polyhedra convex hull algorithm.

added at line 11. Conversely, if  $d_{post}$  is true, the intersection point is degenerate, that is, either  $E$  contains a single inequality or the angular difference between the current inequality and its successor is greater or equal to  $\pi$ . In the example two vertices are created for  $E$ , namely  $v_1$  and  $v_2$  where  $\{v_1\} = \text{intersect}(e_0, e_1)$  and  $\{v_2\} = \text{intersect}(e_1, e_2)$ . The intersection point  $\text{intersect}(e_2, e_0)$  is degenerate, thus it is not added to  $V$ ; in fact the point lies outside the feasible space. Six vertices are created for  $E'$ . Rays are created at line 9 and 10 if the intersection point is degenerate. The two rays along the boundaries of  $e_i$  and  $e_{(i+1) \bmod n}$  are generated in loop iteration  $i$  when  $d_{post}$  is true and iteration  $(i + 1) \bmod n$  when  $d_{pre}$  is true. In our example  $d_{post}$  is true for  $e_2$ , generating a ray along the boundary of  $e_2$  which recedes in the direction of the first quadrant, whereas  $d_{pre}$  is only true for  $e_0$  yielding a ray along  $e_0$  which recedes towards the second quadrant. No rays are created for the polytope  $E'$ .

In general both flags might be true. In this circumstance the current inequality  $e_i$  cannot define a vertex. In this case an arbitrary point on the boundary of the halfspace of  $e_i$  is created at line 12 to fix its representing rays in space. Another case not encountered in this example arises when the polyhedron consists of a single halfspace ( $|E| = 1$ ). In this case a third ray is created (line 4) to indicate on which side the feasible space lies. Note that the maximum number of elements in  $R$  never exceeds four, which occurs when the input defines two facing halfspaces.

The remainder of the *hull* function is dedicated to the reconstruction phase. The point and ray sets, returned by *extreme*, are merged at line 30 and 31. At line 32 the size of a square is calculated which includes all points in  $P$ . The square has  $\langle s, s \rangle$ ,  $\langle -s, s \rangle$ ,  $\langle s, -s \rangle$ ,  $\langle -s, -s \rangle$  as its corners. The square in the running example is depicted in all three frames of Figure 3 and the origin is marked with a cross. Each point  $p \in P$  is then translated by each ray  $r \in R$  yielding the point set  $Q$ . Translated points appear outside the square since all normalised rays are translated by the length of the diagonal  $2\sqrt{2}s$  of the square. The translation process for the worked example is depicted in the second frame. Line 38 is not relevant to this example as it traps the case when the output polyhedron consists of a single point. Line 40 calculates a feasible point  $q_p$  of the convex hull of  $Q$  which is not a vertex. This point serves as the pivot point in the classic Graham scan. First, the point set  $Q$  is sorted counter-clockwise with respect to  $q_p$ . Second, interior points are removed, yielding the indices of all vertices, in the case of the example  $k_0, \dots, k_7$ . What follows is a round-trip around the hull which translates pairs of adjacent vertices into inequalities by calling *connect* at line 48. Whether this inequality actually appears in the result depends on the state of the *add* flag. In our particular example the *add* flag is only set at line 50. Whenever it is set, it is because one of the two vertices lies within the square. The resulting polyhedron consists of the inequalities  $\overline{q_{k_2}, q_{k_3}}$ ,  $\overline{q_{k_3}, q_{k_4}}$  and  $\overline{q_{k_4}, q_{k_5}}$  which is a correct solution for this example.

In general, the reconstruction phase has to consider certain anomalies that mainly arise in outputs of lower dimensionality. One subtlety in the two dimensional case is the handling of polyhedra which contain lines. This is illustrated in Figure 4 where the two inequalities  $e_0, e_1$  are equivalent to one equation which

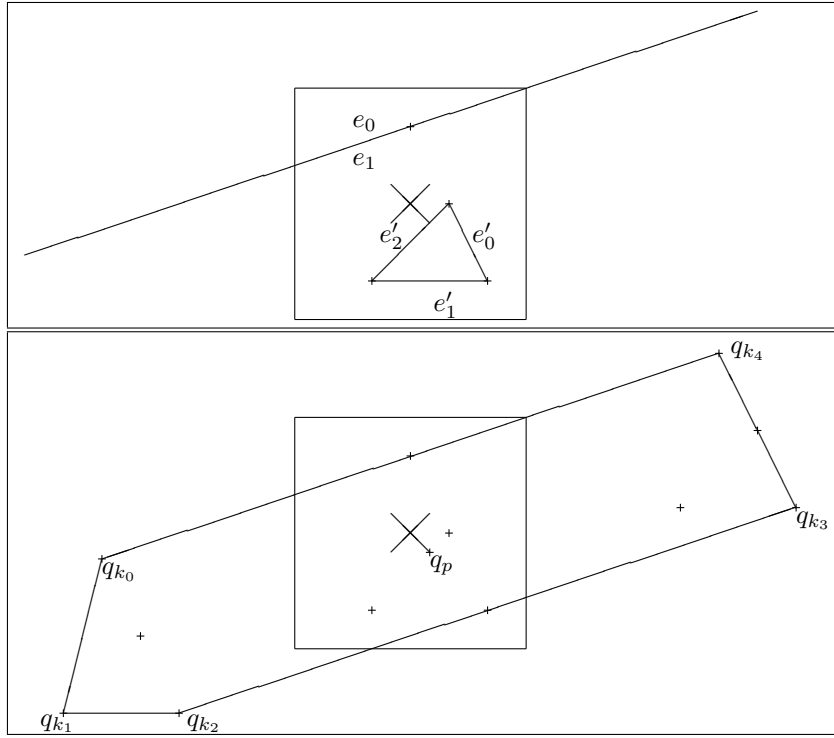


Figure 4: The resulting convex hull contains a line.

defines a space that is a line. Observe from the second frame that no point in the square is a vertex in the hull of  $Q$ . Therefore the predicate *inBox* does not hold for the two vertices  $q_{k_2}$  and  $q_{k_3}$  and the desired inequality  $\overline{q_{k_2}, q_{k_3}}$  is not emitted. Similarly for the vertices  $q_{k_4}$  and  $q_{k_0}$ . However, in such cases there always exists a point in  $p \in Q$  with  $\overline{q_p, q_{k_i}} \angle \overline{q_p, p} < \overline{q_p, q_{k_i}} \angle \overline{q_p, q_{k_{(i+1) \bmod m}}}$  which lies in the square. Thus it is sufficient to search for an index  $j \in [k_i + 1, k_{(i+1) \bmod m} - 1]$  such that  $q_j$  is both in the square and on the line connecting the vertices  $q_{k_i}$  and  $q_{k_{(i+1) \bmod m}}$ . The inner loop at lines 52–56 tests if  $Q$  contains such a point and sets *add* appropriately.

The one dimensional case is handled by the  $m = 2$  tests at line 50 and 57. Figure 5 illustrates why the test in line 50 is necessary. Suppose  $E_1$  and  $E_2$  are given such that  $\text{extreme}(E_1) = \langle \{q_4, q_5\}, \emptyset \rangle$  and  $\text{extreme}(E_2) = \langle \{q_3\}, \{r, -r\} \rangle$  where  $r$  is any ray parallel to the line. Observe that all points are collinear, thus the pivot point is on the line and a stable sort could return the ordering depicted in the figure. The correct inequalities for this example are  $E_{\text{res}} = \{\overline{q_0, q_8}, \overline{q_8, q_0}\}$ . The Graham scan will identify  $q_{k_0} = q_0$  and  $q_{k_1} = q_8$  as vertices. Since there exists  $j \in [k_0 + 1, k_1 - 1]$  such that  $\text{inBox}(s, q_j)$  holds,  $\overline{q_0, q_8} \in E_{\text{res}}$ . In contrast, although there are boundary points between  $q_8$  and  $q_0$  the loop cannot locate

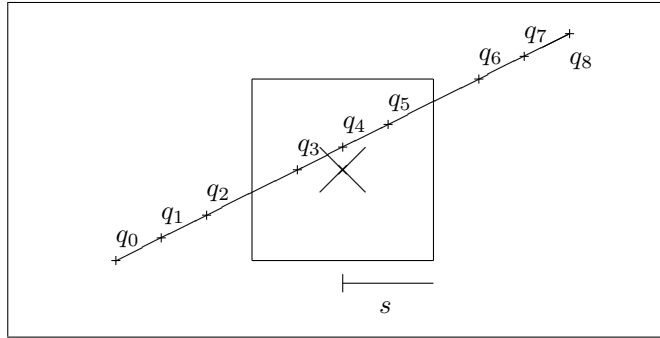


Figure 5: Handling the one-dimensional case.

them due to the ordering. In this case the  $m = 2$  test ensures that  $add$  is set, guaranteeing that  $\overline{q_8, q_0} \in E_{res}$ .

The output polyhedron must include  $q_{k_i}$  as a vertex whenever  $inBox(s, q_{k_i})$  holds. If  $inBox(s, q_{k_i})$  holds, the algorithm generates  $e_{i-1} = \overline{q_{k_{(i-1) \bmod m}}, q_{k_i}}$  and  $e_i = \overline{q_{k_i}, q_{k_{(i+1) \bmod m}}}$ . If  $e_{i-1} \angle e_i < \pi$ , then  $\{q_{k_i}\} = intersect(e_{i-1}, e_i)$  and the vertex  $q_{k_i}$  is realized. Observe that if  $m = 2$ ,  $e_{i-1} \angle e_i = \pi$  which necessitates an additional inequality to define the vertex  $q_{k_i}$ . This is the rôle of the inequality generated on lines 58 or 59. Observe that this inequality  $e$  guarantees  $e_{i-1} \angle e < \pi$  and  $e \angle e_i < \pi$  which is sufficient to define  $q_{k_i}$ .

The zero dimensional case corresponds to the case of when both input polyhedra consist of the single point  $v$ . Line 38 traps this case and returns a set of inequalities describing  $\{v\}$ . Observe that the zero and one dimensional case only require minute changes to the general two dimensional case.

As a note on implementation, observe that the search for a pivot point at line 40 can be refined. One method for finding a definite vertex is to search for a point with extremal coordinates [1]. However, this process requires all points to be examined. The presented algorithm follows Graham [4] in creating an interior point as the pivot point. This does not necessarily require the whole point set to be examined. By choosing two arbitrary points  $q_1, q_2$ , it is sufficient to search the point set for a point  $q_i$  which does not saturate the line  $\overline{q_1, q_2}$ . The centre of the triangle  $q_1, q_2, q_i$  is guaranteed to be an interior point of  $Q$ .

Note also that the sorting in *extreme* is unnecessary if the input inequalities are consecutive in terms of angle. In particular, the output of one run of the algorithm can serve as an input to another without applying the sort at line 2.

Finally observe that the inner loop at lines 52–56 can often be skipped: if the line between  $q_{k_i}$  and  $q_{k_{(i+1) \bmod m}}$  does not intersect with the square,  $inBox(s, q)$  cannot hold for any  $q \in Q$ . Hence  $add$  will not be set at line 54 and the inner loop has no effect.



### 3 Proof of Correctness

Section 3.1 introduces the mathematical language necessary for expressing the two parts of the proof. The proof itself reflects the structure of the algorithm: Section 3.2 concerns the conversion of planar  $H$ -polyhedra into their ray and point representations; Section 3.3 argues that the reconstructed polyhedron encloses the points and rays generated from the two input polyhedra, and yet is also minimal.

#### 3.1 Preliminaries

A polyhedron is a set  $P \subseteq \mathbb{R}^d$  such that  $P = \{\vec{x} \in \mathbb{R}^d \mid A\vec{x} \leq \vec{c}\}$  for some matrix  $A \in \mathbb{R}^{n \times d}$  and vector  $\vec{c} \in \mathbb{R}^n$ . An  $H$ -polyhedron is a pair  $\langle A, \vec{c} \rangle$  where  $A \in \mathbb{R}^{n \times d}$  and  $\vec{c} \in \mathbb{R}^n$  that is interpreted by  $\llbracket \cdot \rrbracket$  as the polyhedron  $\llbracket \langle A, \vec{c} \rangle \rrbracket = \{\vec{x} \in \mathbb{R}^d \mid A\vec{x} \leq \vec{c}\}$ . For brevity we manipulate  $H$ -polyhedra as a finite set of inequalities  $E = \{\vec{a}_1 \cdot \vec{x} \leq c_1, \dots, \vec{a}_n \cdot \vec{x} \leq c_n\}$  which is equivalent to the matrix representation  $\langle A, \vec{c} \rangle$  with  $A = \langle \vec{a}_1, \dots, \vec{a}_n \rangle^T$  and  $\vec{c} = \langle c_1, \dots, c_n \rangle^T$ .  $E$  is said to be satisfiable if  $\llbracket E \rrbracket \neq \emptyset$  and non-redundant if  $\forall e \in E. \llbracket E \setminus \{e\} \rrbracket \neq \llbracket E \rrbracket$ . Two inequalities  $e, e'$  coincide, written  $c(e, e')$ , iff there exists  $\vec{a} \in \mathbb{R}^d, c \in \mathbb{R}$  with  $\llbracket \{e, e'\} \rrbracket = \{\vec{x} \in \mathbb{R}^d \mid \vec{a} \cdot \vec{x} = c\}$ .

The convex hull of a finite set of points  $P = \{\vec{p}_1, \dots, \vec{p}_n\} \subseteq \mathbb{R}^d$  is defined as  $\text{conv}(P) = \{\sum_{i=1}^n \lambda_i \vec{p}_i \mid 0 \leq \lambda_i \wedge \sum_{i=1}^n \lambda_i = 1\}$ . Moreover, the cone of a finite set of vectors  $R = \{\vec{r}_1, \dots, \vec{r}_m\} \subseteq \mathbb{R}^d$  is defined as  $\text{cone}(R) = \{\sum_{i=1}^m \lambda_i \vec{r}_i \mid 0 \leq \lambda_i\}$ . The Minkowski sum of two sets  $X, Y \subseteq \mathbb{R}^d$  is defined as  $X + Y = \{\vec{x} + \vec{y} \mid \vec{x} \in X \wedge \vec{y} \in Y\}$ . Let  $\text{vert}(S) = \{p \in S \mid p \notin \text{conv}(S \setminus \{p\})\}$ ,  $\text{ray}(S) = \{r \in \mathbb{R}^d \setminus \{\vec{0}\} \mid S + \text{cone}(\{r\}) \subseteq S\}$  and  $\text{line}(S) = \{r \in \text{ray}(S) \mid -r \in \text{ray}(S)\}$  denote the vertices, rays and lines of a convex set  $S$ . The following result, accredited to Motzkin [11], relates the two classic representation of polyhedra.

**Theorem 3.1** The following statements are equivalent for any  $S \subseteq \mathbb{R}^d$ :

1.  $S = \text{conv}(P) + \text{cone}(R)$  for some finite  $P, R \subseteq \mathbb{R}^d$ ;
2.  $S = \{\vec{x} \in \mathbb{R}^d \mid A\vec{x} \leq \vec{c}\}$  for some matrix  $A \in \mathbb{R}^{n \times d}$  and vector  $\vec{c} \in \mathbb{R}^n$ .

Our algorithm converts the two (planar) input  $H$ -polyhedra  $E_i$  into their rays  $R_i$  and points  $P_i$  and calculates an  $H$ -polyhedron  $E$  with the smallest  $\llbracket E \rrbracket$  such that  $\llbracket E \rrbracket \supseteq \llbracket E_1 \rrbracket \cup \llbracket E_2 \rrbracket$ . In fact  $\llbracket E \rrbracket = \text{conv}(P_1 \cup P_2) + \text{cone}(R_1 \cup R_2)$ .

#### 3.2 Decomposition

The discussion of the function *extreme* is organised by the dimension of the polyhedron. Reoccurring or self-contained arguments are factored out in the following lemmata. The first lemma states how redundancy can follow from the angular relationship between three inequalities. Since *extreme* requires its input to be non-redundant, useful angular properties of the input inequalities flow from the lemma.

**Lemma 3.1** Suppose  $e_i = a_i x + b_i y \leq c_i$ ,  $i = 1, 2, 3$ . Let  $c_1 = c_3 = 0$ ,  $c_2 \geq 0$ . Then  $\llbracket \{e_1, e_3\} \rrbracket \subseteq \llbracket \{e_2\} \rrbracket$  if  $0 = \theta(e_1) < \theta(e_2) < \theta(e_3) < \pi$ .

**Proof.** To show  $\llbracket \{e_1, e_3\} \rrbracket \subseteq \llbracket \{e_2\} \rrbracket$  it is sufficient to show  $\llbracket \{e_1, e_3\} \rrbracket \subseteq \llbracket \{a_2 x + b_2 y \leq 0\} \rrbracket$ , thus let  $c_2 = 0$ . Furthermore, w.l.o.g. let  $e_1 \equiv x \leq 0$  (since  $\theta(e_1) = 0$ ),  $e_2 \equiv a_2 x + y \leq 0$  (since  $\theta(e_2) < \pi$ ) and  $e_3 \equiv a_3 x + y \leq 0$  (since  $\theta(e_3) < \pi$ ). Note that  $a_2 = \lambda_1 a_1 + \lambda_3 a_3$  and  $b_2 = \lambda_1 b_1 + \lambda_3 b_3$  has the solution  $\lambda_1 = a_2 - a_3$  and  $\lambda_3 = 1$ . Due to  $b_2 = 1$ ,  $a_2 = \cot^{-1}(\theta(e_2))$  and similarly  $a_3 = \cot^{-1}(\theta(e_3))$ . Because  $\theta(e_2) < \theta(e_3)$  and  $\cot$  is an anti-monotone on  $(0, \pi)$ , it follows that  $a_2 > a_3$ , hence  $\lambda_1 > 0$ . Let  $\langle x, y \rangle$  satisfy  $e_1 \equiv x \leq 0$  and  $e_3 \equiv a_3 x + y \leq 0$ . Due to  $\lambda_1 x \leq 0$ ,  $\lambda_1(x) + \lambda_3(a_3 x + y) = (\lambda_1 + a_3)x + y \leq 0 \equiv e_2$ , thus  $e_2$  holds, hence  $\llbracket \{e_1, e_3\} \rrbracket \subseteq \llbracket \{e_2\} \rrbracket$  as required. ■

The following lemma states that there is an injection between vertices and inequalities. This result is used to show that the loop in *extreme* does indeed generate all vertices of the input polyhedron.

**Lemma 3.2** Let  $E = \{e_0, \dots, e_{n-1}\}$  be non-redundant and ordered by  $\theta$  and  $v \in \text{vert}(\llbracket E \rrbracket)$ . Then there exists  $e_i \in E$  such that  $\{v\} = \text{intersect}(e_i, e_{i'})$  and  $e_i \angle e_{i'} < \pi$  where  $i' = (i + 1) \bmod n$ .

**Proof.** Let  $E' \subseteq E$  contain those inequalities that  $v$  saturates. Note that  $|E'| \geq 2$ , in particular there exist  $e, e' \in E'$  with  $e \angle e' \notin \{0, \pi\}$ , otherwise  $\llbracket E' \rrbracket \setminus \{v\}$  is not convex. Choose  $e_i, e_k \in E'$  such that  $e_i \angle e_k < \pi$ . Suppose for the sake of a contradiction that  $k \neq (i + 1) \bmod n$ . Then  $e_j \in E \setminus E'$  exists with  $e_i \angle e_j < e_i \angle e_k$ . W.l.o.g. assume that  $\theta(e_i) = 0$  and  $v = \langle 0, 0 \rangle$ . Then  $e_i \angle e_j < e_i \angle e_k$  reduces to  $0 = \theta(e_i) < \theta(e_j) < \theta(e_k) < \pi$ . Lemma 3.1 implies  $\llbracket E \setminus \{e_j\} \rrbracket \subseteq \llbracket E \rrbracket$  which contradicts the assumption that  $E$  has no redundant inequalities. Thus  $k = (i + 1) \bmod n = i'$  ■

The following result also builds on Lemma 3.1 and complements the previous lemma in that it states when adjacent inequalities have feasible intersection points. The lemma is used to show that *extreme* only generates points in  $\llbracket E \rrbracket$ .

**Lemma 3.3** Let  $E = \{e_0, \dots, e_{n-1}\}$  be satisfiable, non-redundant and ordered by  $\theta$ . For any  $e_i \in E$  if  $e_i \angle e_{(i+1) \bmod n} < \pi$  then  $\text{intersect}(e_i, e_{(i+1) \bmod n}) \subseteq \llbracket E \rrbracket$ .

**Proof.** Let  $e_i \in E$ . Since  $e_i$  is not redundant in  $E$  the boundary of  $e_i$  intersects with the non-empty convex body  $\llbracket E \setminus \{e_i\} \rrbracket$ . Let  $e_m \in E \setminus \{e_i\}$  such that  $\emptyset \neq S = \text{intersect}(e_i, e_m) \subseteq \llbracket E \setminus \{e_i\} \rrbracket$ . Note that if  $|S| > 1$  then  $E = \{e_i, e_m\}$  with  $e_i \angle e_m = \pi$ , hence  $e_i \angle e_{(i+1) \bmod n} < \pi$  never holds. Let  $\{v\} = S$ . It remains to show that  $e_i$  and  $e_m$  are adjacent. Suppose that  $e_i \angle e_m < \pi$  ( $e_m \angle e_i < \pi$  is analogous). Assume for the sake of a contradiction there exists  $e_l \in E \setminus \{e_i\}$  such that  $e_i \angle e_l < e_i \angle e_m$ . W.l.o.g.  $v = \langle 0, 0 \rangle$  and  $\theta(e_i) = 0$ , hence  $0 = \theta(e_i) < \theta(e_l) < \theta(e_m) < \pi$ . Since  $v \in \llbracket E \setminus \{e_i\} \rrbracket$ ,  $v \in \llbracket e_l \rrbracket$  and thus  $c_l \geq 0$  where

$e_l \equiv a_l x + b_l y \leq c_l$ . By Lemma 3.1  $e_l$  is redundant in  $E$  which is a contradiction. It follows that  $m = (i + 1) \bmod n$ . ■

While the previous lemmata concern points, the following lemma is a statement about rays. In particular, it states when inequalities give rise to rays.

**Lemma 3.4** *Let  $E = \{e_0 \equiv a_0 x + b_0 y \leq c_0, \dots, e_{n-1}\}$  be satisfiable, non-redundant and ordered by  $\theta$ . Let  $i' = (i + 1) \bmod n$ ,  $S_1 = \text{cone}(\{\langle -b_i, a_i \rangle, \langle b_{i'}, -a_{i'} \rangle\})$  and  $S_2 = \text{ray}(\llbracket E \rrbracket)$ . If  $e_i \angle e_{i'} > \pi$  or  $|E| \geq 3 \wedge e_i \angle e_{i'} = \pi$  then  $S_1 = S_2$ .*

**Proof.** To show  $S_1 \subseteq S_2$ . Choose  $p \in \llbracket E \rrbracket$  such that  $p$  saturates  $e_i$ . For the sake of a contradiction assume  $r_i = \langle -b_i, a_i \rangle \notin \text{ray}(\llbracket E \rrbracket)$ . Thus there exists  $\lambda > 0$  with  $p + \lambda r_i \notin \llbracket E \rrbracket$ . Set  $\lambda_{\max} = \max\{\lambda > 0 \mid p + \lambda r_i \in \llbracket E \rrbracket\}$ . Note that  $v = p + \lambda_{\max} r_i$  is a vertex and by Lemma 3.2 there exists  $e_k \in E$  with  $\{v\} = \text{intersect}(e_k, e_{(k+1) \bmod n})$  and  $e_k \angle e_{(k+1) \bmod n} < \pi$ . The latter implies that  $k \neq i$ , therefore  $i = (k + 1) \bmod n$ . Hence  $e_{(i-1) \bmod n} = e_k$  does not contain the ray. W.l.o.g. let  $\theta(e_i) = 0$  and  $v = \langle 0, 0 \rangle$ . Then  $r_i = \langle 0, 1 \rangle$  and  $\pi < \theta(e_{(i-1) \bmod n}) < 2\pi$ , hence  $e_{(i-1) \bmod n} \equiv a'x + b'y \leq 0$  with  $b' < 0$ . Thus  $a'x + b'(\lambda + y) \leq 0$ , hence  $p + \lambda r_i \in \llbracket \{e_{(i-1) \bmod n}\} \rrbracket$  which is a contradiction. Analogously for  $r_{i'} = \langle b_{i'}, -a_{i'} \rangle$ . Now to show  $S_2 \subseteq S_1$ . Assume there exists  $r \in \text{ray}(\llbracket E \rrbracket) \setminus \text{cone}(\{\langle -b_i, a_i \rangle, \langle b_{i'}, -a_{i'} \rangle\})$ . Consider  $e_i \angle e_{i'} > \pi$ . Since  $\theta(e_i) \neq \theta(e_{i'})$  there exists  $\lambda_1, \lambda_2 \in \mathbb{R}$  with  $r = \lambda_1 \langle -b_i, a_i \rangle + \lambda_2 \langle b_{i'}, -a_{i'} \rangle$ . Assume  $\lambda_2 < 0$ . W.l.o.g. let  $\{p\} = \text{intersect}(e_i, e_{i'}) = \{\langle 0, 0 \rangle\}$  and  $\theta(e_i) = 0$  thus let  $e_i \equiv x \leq 0$ . It follows that  $\pi < \theta(e_{i'}) < 2\pi$ ,  $b_{i'} \leq 0$  and we set  $e_{i'} \equiv a'_i x - y \leq 0$ . Thus  $r = \lambda_1 \langle 0, 1 \rangle + \lambda_2 \langle -1, -a'_i \rangle = \langle -\lambda_2, \lambda_1 - \lambda_2 a'_i \rangle$ . Since  $\lambda_2 < 0$ ,  $p + r \notin \llbracket \{e_i\} \rrbracket$  which contradicts  $r \in \text{ray}(\llbracket E \rrbracket)$ . Analogously for  $\lambda_1 < 0$  with  $\theta(e_{i'}) = 0$ . Since  $\lambda_1 \geq 0$  and  $\lambda_2 \geq 0$ ,  $r \in \text{cone}(\{\langle -b_i, a_i \rangle, \langle b_{i'}, -a_{i'} \rangle\})$  which is a contradiction. Now suppose  $e_i \angle e_{i'} = \pi$  and  $|E| \geq 3$ . W.l.o.g. let  $\theta(e_i) = 0$ ,  $e_i \equiv x \leq |u|$  and  $e_{i'} \equiv -x \leq 0$ . Thus  $S_1 = \text{cone}(\{\langle 0, 1 \rangle\})$ . Let  $\langle x_r, y_r \rangle = r$  and  $p \in \llbracket \{e_i, e_{i'}\} \rrbracket$ . Observe that  $x_r = 0$  otherwise there exists  $\lambda > 0$  with  $p + \lambda r \notin \llbracket \{e_i, e_{i'}\} \rrbracket$ . Now assume  $y_r < 0$ . Since  $|E| \geq 3$ , there exists  $e_j \in E$  such that  $\pi < \theta(e_j) < 2\pi$  and thus  $e_j \equiv a_j x + b_j y \leq c_j$  with  $b_j < 0$ . Let  $p \in \llbracket \{e_j\} \rrbracket$ , then there exists  $\lambda > 0$  such that  $p + \lambda r \notin \llbracket \{e_j\} \rrbracket$  which is a contradiction. ■

The correctness of the first stage of our algorithm is summarised by the following proposition. Note that the Minkowski sum  $\text{conv}(V) + \emptyset$  always defines the empty space rather than the polyhedron  $\text{conv}(V)$ . Thus a null ray  $\langle 0, 0 \rangle$  is required to represent a bounded polyhedron. The function *extreme* avoids adding this null ray for the sake of improved efficiency. In *hull* the translation of points at line 36 by the null ray is replaced by a simple copying step at line 35. However, the null ray still manifests itself in the correctness results.

**Proposition 3.1** *Let  $E \subset \text{Lin}$  be non-empty, finite, satisfiable and non-redundant. Then  $\text{extreme}(E) = \langle P, R \rangle$  and  $\llbracket E \rrbracket = \text{conv}(P) + \text{cone}(R \cup \{\langle 0, 0 \rangle\})$ .*

Note that the following proof handles polyhedra that contain lines as a special case. This distinction is not artificial, in fact Klee [5] observed that a closed convex set that does not contain lines is generated by its vertices and its extreme rays. (Extreme rays are rays that cannot be expressed by a linear combination of others.) In order to describe polyhedra which contain lines it is necessary to create points that are not vertices and rays which are not extreme.

**Proof.** Let  $\{e_0, \dots, e_{n-1}\} = E$  such that  $\theta(e_0) < \theta(e_1) < \dots < \theta(e_{n-1})$ . Such an ordering exists since if  $\theta(e_i) = \theta(e_j)$  for some  $i \neq j$  then either  $\llbracket e_i \rrbracket \subseteq \llbracket e_j \rrbracket$ , thus  $\llbracket E \setminus \{e_j\} \rrbracket = \llbracket E \rrbracket$  or vice-versa which contradicts that  $E$  is non-redundant. Let  $\langle P, R \rangle = \text{extreme}(E)$ .

- Suppose  $\text{line}(\llbracket E \rrbracket) \neq \emptyset$ . Note that if there exist  $e_i, e_j \in E$  with  $e_i \angle e_j < \pi$  then  $\text{line}(\llbracket E \rrbracket) = \emptyset$ , hence  $E = \{e\}$  or  $E = \{e_0, e_1\} \wedge e_0 \angle e_1 = \pi$ .
  - Consider  $E = \{e\}$ . The loop is executed only once with  $d_{pre} = d_{post} = \text{true}$ , thus the boundary point  $p$  and two rays  $r_1, r_2$  are added in line 12, 9 and 10, respectively. The boundary of  $\llbracket E \rrbracket$  is  $\{p + \lambda_1 r_1 + \lambda_2 r_2 \mid \lambda_i \geq 0\}$ . The ray  $r_3$  added in line 4 points into the halfspace, thus  $\llbracket E \rrbracket = \{p\} + \text{cone}(\{r_1, r_2, r_3\})$ .
  - Consider  $E = \{e_0, e_1\} \wedge e_0 \angle e_1 = \pi$ . For each  $e_i$ ,  $d_{pre} = d_{post} = \text{true}$ , hence for each  $e_i$  the loop generates two rays  $r_i, r'_i$  along the boundary of  $e_i$  (lines 9, 10) and one boundary point  $p_i$  (line 11). The set  $\text{conv}(\{p_1, p_2\})$  is included in  $\llbracket E \rrbracket$ . Note that the rays generated in the second iteration are collinear to those in the first. Thus  $\llbracket E \rrbracket = \text{conv}(\{p_1, p_2\}) + \text{cone}(\{r_1, r'_1\})$ .
- Suppose  $E$  is zero dimensional, hence  $\llbracket E \rrbracket = \{v\}$ . Let  $\alpha_i = e_i \angle e_{(i+1) \bmod n}$ . Since  $\sum_{i=0}^{n-1} \alpha_i = 2\pi$ , it follows that  $n \geq 3$  since otherwise there exists  $i$  such that  $\alpha_i \geq \pi$  and by Lemma 3.4 it follows that  $\text{ray}(\llbracket E \rrbracket) \neq \emptyset$ . By Lemma 3.1, it follows that  $\alpha_i + \alpha_{(i+1) \bmod n} \geq \pi$ , which for all  $e_i$  is  $\sum_{i=0}^{n-1} (\alpha_i + \alpha_{(i+1) \bmod n}) \geq n\pi$ . However,  $n\pi \leq \sum_{i=0}^{n-1} (\alpha_i + \alpha_{(i+1) \bmod n}) = 2 \sum_{i=0}^{n-1} \alpha_i = 4\pi$ , hence  $n \leq 4$ .
  - Consider  $E = \{e_0, e_1, e_2, e_3\}$ . Observe that for all  $0 \leq i \leq 3$ ,  $\alpha_i + \alpha_{(i+1) \bmod n} = \pi$  and  $\text{intersect}(e_i, e_{(i+1) \bmod n}) = \{v\}$ , hence  $c(e_0, e_2) \wedge c(e_1, e_3)$ . In all loop iterations  $d_{pre} = d_{post} = \text{false}$  thus only vertex  $\{v\} = P$  is generated (line 11).
  - Consider  $E = \{e_0, e_1, e_2\}$ . Observe that for all  $i$ ,  $e_i \angle e_{(i+1) \bmod n} < \pi$ , otherwise by Lemma 3.4,  $\text{ray}(\llbracket E \rrbracket) \neq \emptyset$ . Thus in each iteration  $i$ ,  $d_{post} = d_{pre} = \text{false}$  and  $P = \{v\} = \text{intersect}(e_i, e_{(i+1) \bmod n})$  is generated. Hence  $\llbracket E \rrbracket = \{v\} + \text{cone}(\{(0, 0)\})$ .
- Suppose  $E$  is one dimensional and  $\text{line}(\llbracket E \rrbracket) = \emptyset$ . Since the boundary of  $\llbracket E \rrbracket$  contains infinitely many points and  $|E| = n$  (which is finite), there exists  $e \in E$  such that  $p_1, p_2 \in \llbracket E \rrbracket$  where  $p_1 \neq p_2$  and  $p_1$  and  $p_2$  saturate  $e$ . In fact there are infinitely many boundary points on the line between

$p_1$  and  $p_2$  which saturate  $e$ . Observe  $\llbracket E \rrbracket$  contains no interior points, hence  $\llbracket E \rrbracket$  consists only of boundary points. Therefore there exists  $e' \in E$  that saturates infinitely many of these points and for which  $c(e, e')$  holds. As we assume that  $\text{line}(\llbracket E \rrbracket) = \emptyset$ ,  $|E| < 2$ . Due to non-redundancy, there exists at most one  $e_i \in E$  with  $0 < e \angle e_i < \pi$ . Similarly for  $e'$ . Hence  $3 \leq |E| \leq 4$  follows.

- Consider  $E = \{e_0, e_1, e_2\}$ . Let  $i \in [0, 2]$  such that  $c(e_i, e_{(i+1) \bmod n})$  holds. On iteration  $i$  the loop generates a ray  $r$  along the boundary of  $\llbracket e_i \rrbracket$  in line 10. A collinear ray is generated in iteration  $(i+1) \bmod n$  for  $e_{(i+1) \bmod n}$  on line 9. It is in this and iteration  $(i+2) \bmod n$  where  $\{v\} = \text{intersect}(e_{(i+1) \bmod n}, e_{(i+2) \bmod n}) = \text{intersect}(e_{(i+2) \bmod n}, e_i)$  is added to  $P$ . Thus  $\llbracket E \rrbracket = \{v\} + \text{cone}(\{r\})$ .
- Consider  $E = \{e_0, e_1, e_2, e_3\}$ . Let  $i \in [0, 3]$  such that  $c(e_i, e_{(i+2) \bmod n})$  holds. In all four iterations  $d_{pre} = d_{post} = \text{false}$  holds, resulting in two vertices resulting from the intersection of adjacent inequalities:  $\{v_1\} = \text{intersect}(e_i, e_{(i+1) \bmod n}) = \text{intersect}(e_{(i+1) \bmod n}, e_{(i+2) \bmod n})$ ,  $\{v_2\} = \text{intersect}(e_{(i+2) \bmod n}, e_{(i+3) \bmod n}) = \text{intersect}(e_{(i+3) \bmod n}, e_i)$ . Hence  $\llbracket E \rrbracket = \text{conv}(\{v_1, v_2\}) + \text{cone}(\{(0, 0)\})$ .
- Suppose that  $E$  is two dimensional and that none of the preceding cases apply.
  - Since  $c$  never holds for  $|E| > 4$  the previous cases deal with all  $e, e' \in E$  where  $c(e, e')$  holds. Because  $\llbracket E \rrbracket$  does not consist of a single point described by three inequalities, it must be two dimensional.
  - Let  $v \in V = \text{vert}(\llbracket E \rrbracket)$ . By Lemma 3.2 there exists  $e_i \in E$  such that  $e_i \angle e_{(i+1) \bmod n} < \pi$  and  $\{v\} = \text{intersect}(e_i, e_{(i+1) \bmod n})$ . Then  $d_{post}$  is false in iteration  $i$ , hence  $v \in P$ , thus  $V \subseteq P$ . By Lemma 3.3  $P \subseteq \llbracket E \rrbracket$ . If for all  $i$ ,  $e_i \angle e_{(i+1) \bmod n} < \pi$  holds then  $\llbracket E \rrbracket = \text{conv}(P) + \text{cone}(\langle 0, 0 \rangle)$ . Otherwise let  $i \in [0, m-1]$  such that  $e_i \angle e_{(i+1) \bmod n} \geq \pi$ . Then  $d_{post}$  will be true in iteration  $i$  and  $d_{pre}$  will hold in iteration  $(i+1) \bmod n$  yielding rays that are collinear to  $r_1 = \langle -b_i, a_i \rangle$  and  $r_2 = \langle b_{(i+1) \bmod n}, -a_{(i+1) \bmod n} \rangle$ . Since the previous cases do not apply, whenever  $e_i \angle e_{(i+1) \bmod n} = \pi$ ,  $|E| \geq 3$  hence Lemma 3.4 can be applied. It follows that  $\text{ray}(\llbracket E \rrbracket) = \text{cone}(\{r_1, r_2\})$ , hence  $\llbracket E \rrbracket = \text{conv}(P) + \text{cone}(\{r_1, r_2\})$ .

■

### 3.3 Reconstruction

One advantage of the point and ray representation (over one that makes lines explicit) is that *extreme* naturally generates lines as two opposing rays in independent iterations, thus an explicit line case is not required. The remainder of

the *hull* function combines the points and rays of the two input polyhedra to construct a corresponding set of inequalities. The advantage of the simplified representation carries over to the reconstruction phase in that opposing rays from different polyhedra need not be recognised and reconstituted as a line.

**Theorem 3.2** *Given  $E_1, E_2 \subset \text{Lin}$  be non-empty, finite, satisfiable and non-redundant,  $E_{\text{res}} = \text{hull}(E_1, E_2)$  is non-redundant and the smallest  $E_{\text{res}} \subset \text{Lin}$  with  $\llbracket E_{\text{res}} \rrbracket \supseteq \llbracket E_1 \rrbracket \cup \llbracket E_2 \rrbracket$  such that for all  $E \in \text{Lin}$ ,  $\llbracket E \rrbracket \supseteq \llbracket E_1 \rrbracket \cup \llbracket E_2 \rrbracket \Rightarrow \llbracket E_{\text{res}} \rrbracket \subseteq \llbracket E \rrbracket$ .*

**Proof.** The case for  $E_1 = \emptyset$  or  $E_2 = \emptyset$  on line 27 is trivial, thus assume  $E_1 \cup E_2 \neq \emptyset$  and that lines 28–35 are executed. Note that for all  $r \in R$ ,  $|r| = 1$  due to the normalisation in *extreme*. Thus  $(P + \{2\sqrt{2}sr \mid r \in R\}) \cap (-s, s)^2 = \emptyset$ . Hence after line 12 the predicate *inBox*( $s, p$ ) holds whenever  $p \in P = Q \setminus (P + \{2\sqrt{2}sr \mid r \in R\})$ . Line 38 handles the zero dimensional case, hence from line 39 onwards  $|Q| > 1$ . The arithmetic mean  $q_p$  of all points is calculated in line 40. This point serves as reference when comparing two points for counter-clockwise ordering. Observe that  $q_p \in \text{conv}(Q)$ , in particular  $q_p$  is not on its boundary if  $\text{conv}(Q)$  is two dimensional. The latter ensures for all boundary points  $q_1, q_2 \in \text{conv}(Q)$ ,  $\theta(\overline{q_p, q_1}) \neq \theta(\overline{q_p, q_2})$ , thus line 18 yields a total ordering on the boundary points in the two dimensional case. Line 44 performs the classic Graham scan which identifies the vertices of  $\text{conv}(Q)$ .

- To show  $\llbracket E_{\text{res}} \rrbracket \supseteq \llbracket E_1 \rrbracket \cup \llbracket E_2 \rrbracket$ . In particular, show  $\llbracket E_{\text{res}} \rrbracket \supseteq \llbracket E_1 \rrbracket$ , i.e. for all  $e \in E_{\text{res}}$ ,  $\llbracket \{e\} \rrbracket \supseteq \llbracket E_1 \rrbracket$ .
  - To show  $\text{conv}(Q) \subseteq \llbracket \{e\} \rrbracket$ . Suppose  $e$  is added in line 60. Then the flag *add* was true and  $e = \text{connect}(q_{k_i}, q_{k_{(i+1) \bmod m}})$ . For the sake of a contradiction, suppose there exists  $q_{k_j} \notin \llbracket \{e\} \rrbracket$  such that  $j \notin \{i, (i+1) \bmod m\}$ . W.l.o.g. let  $\theta(\overline{q_p, q_{k_j}}) < \theta(\overline{q_p, q_{k_i}})$ . There exists a line through  $q_p$  and  $q_{k_j}$  which intersects the boundary of  $e$  at a point  $z$ . Then  $q_{k_j} \in \text{conv}(\{z, q_{k_{(i+1) \bmod m}}\})$  which contradicts that  $q_{k_i}$  is a vertex. Since all  $q_{k_0}, \dots, q_{k_{m-1}} \in \llbracket \{e\} \rrbracket$  it follows that  $\text{conv}(Q) \subseteq \llbracket \{e\} \rrbracket$ . Furthermore it is easy to verify that the inequality  $e$  added in line 58 or 59 holds for both  $\{q_{k_0}, q_{k_1}\} = Q$ , hence  $\text{conv}(Q) \subseteq \llbracket \{e\} \rrbracket$ .
  - To show  $\text{cone}(R) \subseteq \text{ray}(\llbracket \{e\} \rrbracket)$ .
    - \* Suppose  $m > 2$ . If either *add* was set at line 50 or line 54, there exists  $q_j \in Q$  which saturates  $e$  such that *inBox*( $s, q_j$ ) holds, hence  $q_j \in P$ . Let  $r \in R$ . Then  $q_j + 2\sqrt{2}sr \in \text{conv}(Q) \subseteq \llbracket \{e\} \rrbracket$ . Since  $q_j$  saturates  $e$ , it follows that  $r \in \text{ray}(\llbracket \{e\} \rrbracket)$ .
    - \* Suppose  $m = 2$ . Assume that *inBox*( $s, q_{k_0}$ ) and *inBox*( $s, q_{k_1}$ ) both do not hold, then there exist  $p \in P$  and  $r \in R$  with  $q_{k_0} = p + 2\sqrt{2}sr$ . Note that since  $m = 2$ ,  $p$  saturates  $e$ . Continue as above. Assume  $e$  was added in lines 58–59. Observe that  $q_j = q_{k_i}$  saturates  $e$ . Again, continue as in the first case.

Thus  $\llbracket E_1 \rrbracket = \text{conv}(P_1) + \text{cone}(R_1) \subseteq \text{conv}(P) + \text{cone}(R) \subseteq \llbracket \{e\} \rrbracket$ . Similarly for  $E_2$ . Thus  $\llbracket E_{\text{res}} \rrbracket \supseteq \llbracket E_1 \rrbracket \cup \llbracket E_2 \rrbracket$ .

- To show for all  $E \subset \text{Lin}$ ,  $\llbracket E \rrbracket \supseteq \llbracket E_1 \rrbracket \cup \llbracket E_2 \rrbracket \Rightarrow \llbracket E_{\text{res}} \rrbracket \subseteq \llbracket E \rrbracket$ . For the sake of a contradiction suppose there exists  $p \in \llbracket E_{\text{res}} \rrbracket$  such that  $p \notin \text{conv}(P) + \text{cone}(R)$ . Hence for all  $e \in E_{\text{res}}$ ,  $p \in \llbracket \{e\} \rrbracket$ . Let  $p' \in \text{conv}(P) + \text{cone}(R)$  such that  $|p - p'|$  is minimal. Observe that  $p'$  is unique due to convexity.
  - Suppose  $p' \in \text{vert}(\text{conv}(P) + \text{cone}(R))$ . Hence there exists  $i \in [0, m - 1]$  such that  $p' = q_{k_i}$ . Assume that  $\text{inBox}(s, q_{k_i})$  does not hold. Then there exists  $p'' \in P$ ,  $r \in R$  and  $\lambda > 0$  such that  $q_{k_i} = p'' + \lambda r$ . Since  $\llbracket E_{\text{res}} \rrbracket \supseteq \text{conv}(P) + \text{cone}(R)$ ,  $p'' + 2\lambda r \in \llbracket E_{\text{res}} \rrbracket$ , thus  $q_{k_i} \in \text{conv}(\{p'', p'' + 2\lambda r\})$  which is a contradiction, hence  $\text{inBox}(s, q_{k_i})$  holds. Thus the flag *add* is set on line 54 in loop iteration  $(i - 1) \bmod m$  and  $i$ , hence the inequalities  $e_{(i-1) \bmod m} = \overline{q_{k_{(i-1) \bmod m}}, q_{k_i}}$  and  $e_i = \overline{q_{k_i}, q_{k_{(i+1) \bmod m}}}$  are added to  $E_{\text{res}}$ .
    - \* Suppose  $e_{i-1} \angle e_i < \pi$ . Then  $\{q_{k_i}\} = \text{intersect}(e_{i-1}, e_i)$ . Due to convexity  $|q_{k_{(i-1) \bmod m}} - p| > |q_{k_i} - p|$  and  $|q_{k_{(i+1) \bmod m}} - p| > |q_{k_i} - p|$ . Hence  $p' = q_{k_i}$  is the closest point to  $p$  in the space  $\llbracket \{e_{(i-1) \bmod m}, e_i\} \rrbracket$ . But this implies that  $p \notin \llbracket E_{\text{res}} \rrbracket$  which is a contradiction.
    - \* Suppose  $e_{i-1} \angle e_i = \pi$ . Then  $q_{k_{(i-1) \bmod m}} = q_{k_{(i+1) \bmod m}}$ , thus  $m = 2$ . Note that  $p \in \llbracket \{e_{i-1}, e_i\} \rrbracket$  otherwise  $p \notin E_{\text{res}}$ . Since  $q_{k_i} \in P$ ,  $\text{inBox}(s, q_{k_i})$  holds and line 58 or 59 adds an inequality  $e \in E_{\text{res}}$  with  $\{q_{k_i}\} = \text{intersect}(e_{i-1}, e) = \text{intersect}(e, e_i)$ . Observe that  $p \notin \llbracket \{e\} \rrbracket$  otherwise  $p \in \text{conv}(\{q_{k_i}, q_{k_{(i+1) \bmod m}}\})$ , thus  $p \notin \llbracket E_{\text{res}} \rrbracket$ .
  - Suppose  $p' \notin \text{vert}(\text{conv}(P) + \text{cone}(R))$ , thus  $p'$  is a boundary point of  $\text{conv}(P) + \text{cone}(R)$ . There exists a line  $L \subset \mathbb{R}^2$  such that  $p' \in L$  and  $\text{conv}(P) + \text{cone}(R) \setminus L$  is convex (but not closed). There exists a loop iteration  $i$  such that the boundary of  $e = \text{connect}(q_{k_i}, q_{k_{(i+1) \bmod m}})$  is exactly  $L$ . Since  $\text{conv}(P) + \text{cone}(R) \subseteq \llbracket \{e\} \rrbracket$ ,  $p \notin \llbracket \{e\} \rrbracket$ .
    - \* Suppose  $\text{inBox}(s, q_{k_i})$  or  $\text{inBox}(s, q_{k_{(i+1) \bmod m}})$  holds. The flag *add* is true, thus  $e \in E_{\text{res}}$  and  $p \notin \llbracket E_{\text{res}} \rrbracket$ .
    - \* Suppose neither  $\text{inBox}(s, q_{k_i})$  nor  $\text{inBox}(s, q_{k_{(i+1) \bmod m}})$  holds. There exists  $q_j \in P$  such that  $q_{k_i} = q_j + \lambda r$  for some  $\lambda > 0$  and  $r \in R$ . Due to the ordering of the points,  $k_i < j < k_{(i+1) \bmod m}$  whenever  $\text{conv}(P) + \text{cone}(R)$  is two dimensional. Thus *add* is set in line 54. If  $\text{conv}(P) + \text{cone}(R)$  is one dimensional,  $m = 2$  and *add* is set in line 49. In both cases  $e \in E_{\text{res}}$  and thus  $p \notin \llbracket \{e\} \rrbracket$ .

It remains to show that  $E_{\text{res}}$  is non-redundant. Note that the loop at lines 46–62 iterates once for each vertex  $q_{k_i}$ , creating inequalities  $e_i = \overline{q_{k_i}, q_{k_{(i+1) \bmod m}}}$ . For all  $i \neq j$ ,  $\theta(e_i) \neq \theta(e_j)$  due to the fact that  $q_{k_0}, \dots, q_{k_{m-1}}$  are consecutive vertices of the hull of  $\text{conv}(Q)$ , thus  $\{e_0, \dots, e_{m-1}\}$  has no redundancies. Now consider the inequality  $e'_i$  added at line 58 or 59. Since  $m = 2$ ,  $e_i \angle e_{(i+1) \bmod m} =$

$\pi$  and since  $\theta(e_i) < \theta(e'_i) < \theta(e_{(i+1) \bmod m})$ , the inequality  $e'_i$  is not redundant. Hence  $E_{\text{res}} \subseteq \{e_0, e'_0, \dots, e_{m-1}, e'_{m-1}\}$  is non-redundant. ■

The running time of the algorithm is  $O(n \log n)$  where  $n = |E_1| + |E_2|$ . After the sorting step at line 2 in *extreme*, each inequality generates at most two rays and one point. Thus *extreme* is in  $O(n \log n)$ . The flag  $d_{pre}$  is true if the angle between two consecutive inequalities is at least  $\pi$ . Thus  $d_{pre}$  can only be true in at most two loop iterations. Similarly for  $d_{post}$ . Hence *extreme* returns at most four rays for each polyhedron, thus  $|R| \leq 8$  at line 31 and  $O(|Q|) = O(n)$ . The dominating cost in the *hull* function is the sorting step at line 42 which we assume is in  $O(n \log n)$ . The scan is linear [4] and partitions the point set  $Q$  into vertices and non-vertices. The loop at lines 46–62 runs once for each vertex, whereas the inner loop at lines 52–56 runs at most once for each non-vertex. It follows that the overall running time is in  $O(n \log n)$ .

## 4 Conclusion

An  $O(n \log n)$  algorithm for calculating the convex hull of planar  $H$ -polyhedra has been presented, thereby improving on existing approaches. The algorithm applies a novel box construction which reduces the problem to calculating the convex hull of a set of points. Implementing the algorithm exposed a number of subtleties which motivated a complete proof of the algorithm.

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