Inferring polynomial invariants

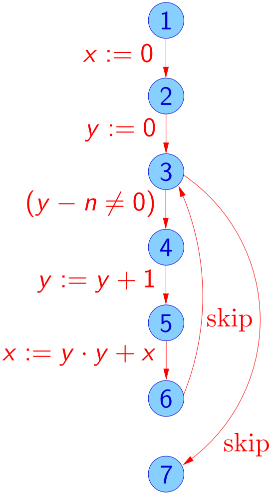
with Polyinvar

Helmut Seidl and Michael Petter

TU-München

Chair Workshop, 2005

Problem:



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Is $x = y$ valid at program point 3?

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What relation holds at program point 7?

$\Rightarrow$ *Polynomial invariants*

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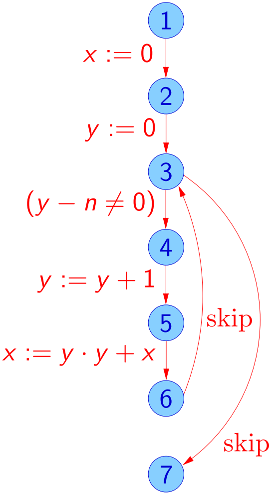

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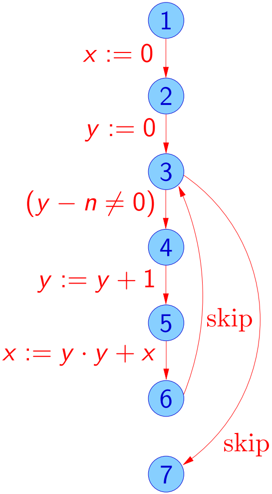

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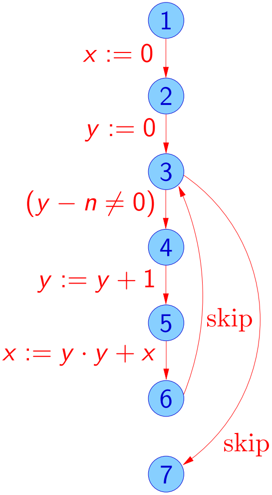

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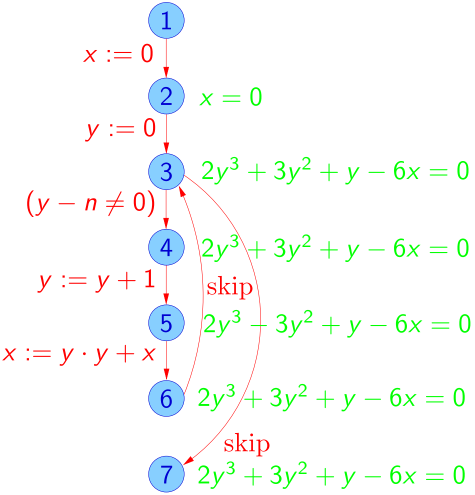
Valid invariants:



Power sum

The example program calculates the square power sum $x = \sum_{y=0}^{n} y^2$, therefore

$$x = \frac{2y^3 + 3y^2 + y}{6}$$

holds at program point 7

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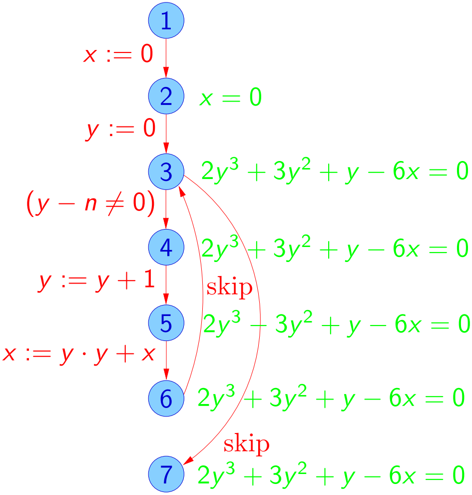

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Related work

Approaches with ideals

E.RC, D.K. Program Verification Using Automatic Generation Of Invariants 2004

M.MO, H.S. Computing Polynomial Program Invariants 2004

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Approach with modules

M.P. Berechnung von polynomiellen Invarianten 2004

Initial point

Interpret program states as Ideals of polynomials;
Store generators of the ideal as representation → M.MO., H.S.

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Abstract Model

Polynomial programs...

- modelling control flow with (possibly annotated) edges
- assignments of multivariate polynomial expressions (without division) $x := y \cdot y + x$
- method calls $x := f(y, z)$
- unknown assignments $x :=?$

... with guards

- negative polynomial equality guards $(y - n) \neq 0$
- positive polynomial equality guards $(y - n) = 0$
- non deterministic choice for the rest *skip*

→ **Goal**: inferring all valid polynomial relations

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Intraprocedural example

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squarepowsum (n ∈ ℕ) ∈ ℕ {
  x, y ∈ ℕ;
  x ⇐ 0, y ⇐ 0;
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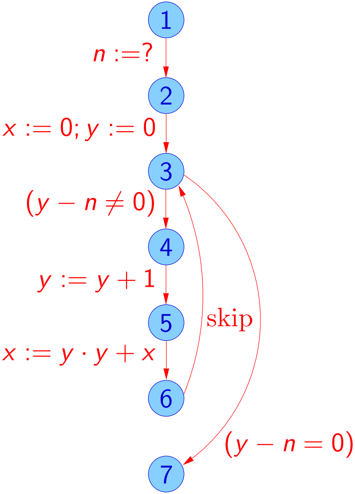

State abstraction

Still, we have to find an abstraction for program states that serves our analysis...

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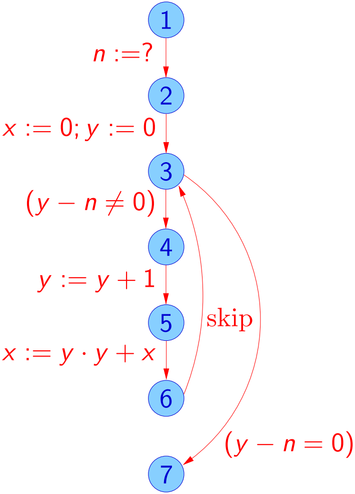


State abstraction

Polynomials

Polynomials are expressed by equations from the set $\mathbb{R}[\mathbf{X}]$, polynomials over $\mathbb{R}$ and the variables from $\mathbf{X}$, for example $x - y^2 + 25 = 0$.

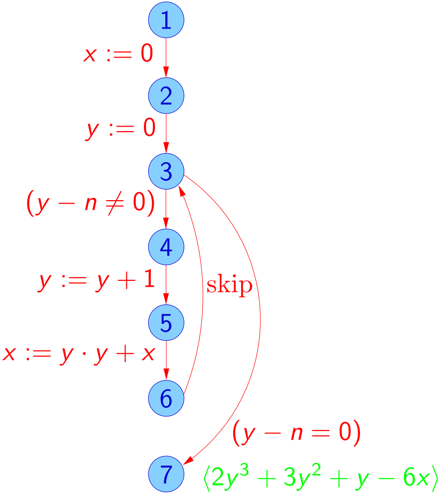
Polynomial relations

1. $\forall_{PR[s] \subseteq \mathbb{R}[\mathbf{X}]} \quad \exists_{p \in PR[s]} \quad \forall_{c \in \mathbb{R} \cup \mathbf{X}} \Rightarrow c \cdot p \in PR[s]$
2. $\forall_{PR[s] \subseteq \mathbb{R}[\mathbf{X}]} \quad \exists_{p \in PR[s]} \quad \forall_{q \in PR[s], \circ \in \{+,-,\cdot\}} \Rightarrow q \circ p \in PR[s]$

Polynomial ideals – finitely generated

Polynomial ideals are infinite sets of polynomials, with the upper properties. All ideals can be represented by a minimal number of generating polynomials.
For example $\langle \{x - y^2 + 25, x^2 - z\} \rangle$

Verifying polynomial relations



Fixpoint analysis

Associating program states with polynomial ideals.

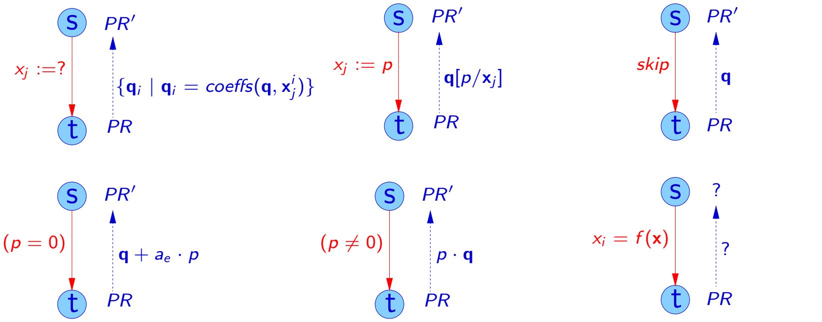
Verifing polynomials

Computing the weakest precondition for a polynomial invariant ideal

Weakest precondition

The only valid precondition can only be the relation $0 = 0$.

Incremental fixpoint iteration: semantics



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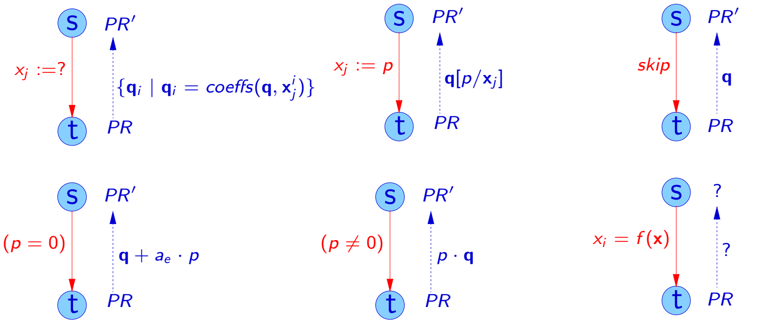

Propagate only new generators – incremental iteration

Recalculation of ideals at each iteration step is expensive
$\Rightarrow$ Only new generators $\mathbf{q}$ have to be propagated via edges.

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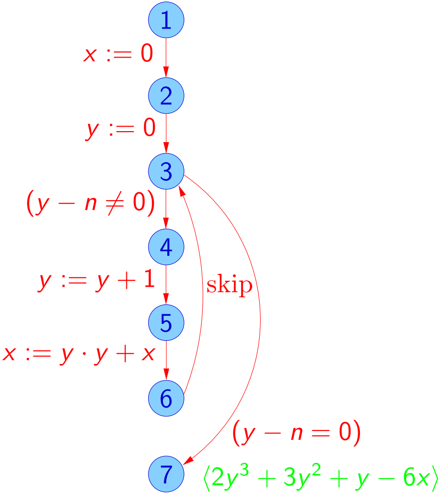

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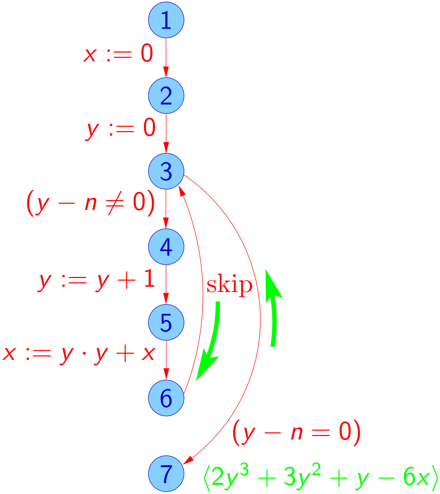

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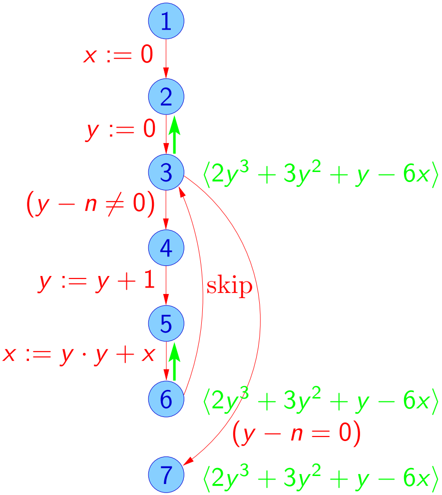

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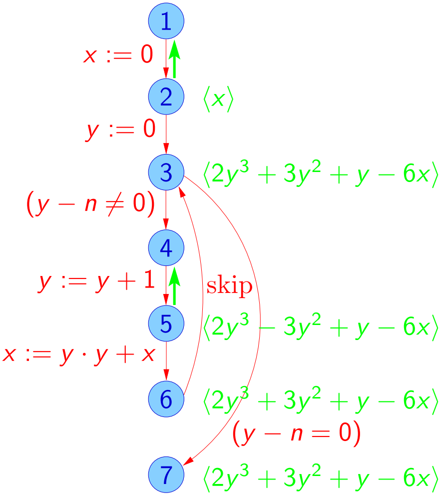

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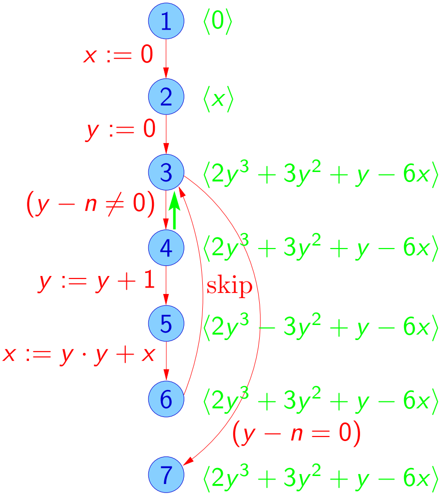

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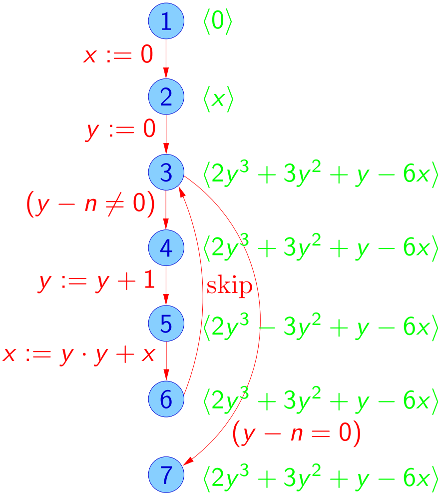

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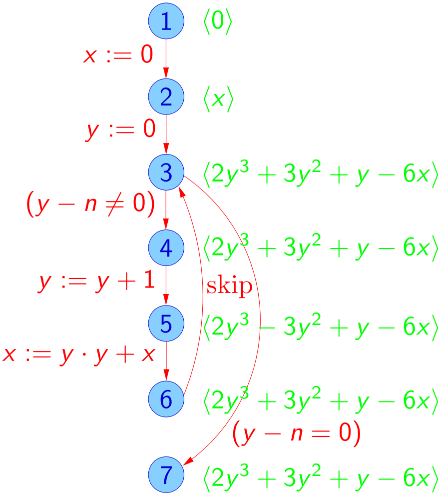

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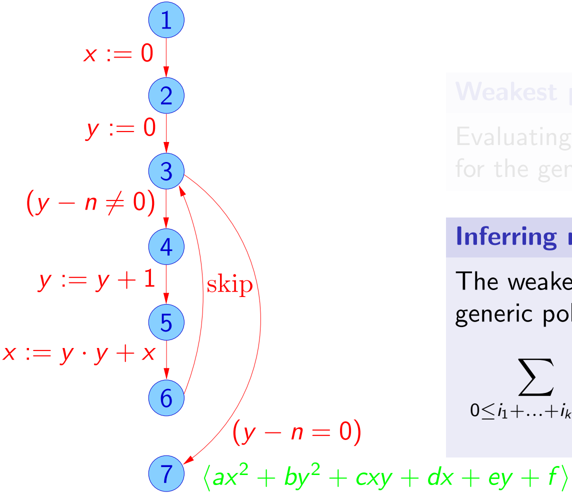

Weakest precondition

Evaluating the WP provides values for the generic parameters

Inferring relations

The weakest precondition for a generic polynomial of degree n. E.g:

$$\sum_{0 \leq i_1 + \ldots + i_k \leq d} \mathbf{a}_{i_1,\ldots,i_k} \cdot \mathbf{x_1}^{i_1} \cdot \ldots \cdot \mathbf{x_k}^{i_k}$$

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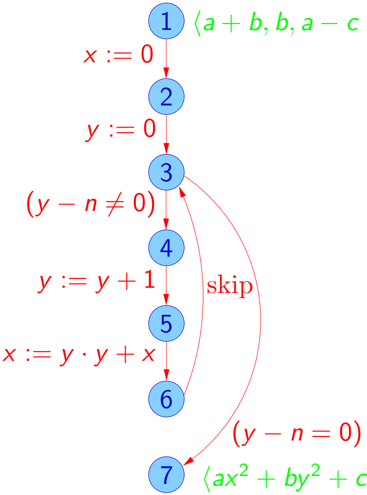

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Performance issues

Bad news

- Reductions on polynomial ideals are perform doubly exponentially on the number of participating variables.
- Ideal membership is in general EXPSPACE-hard
- Ideal membership is NP-hard for fixed number of variables

▽ Problem

Using generic polynomials with many variables turns polynomial reductions infeasible.

⇒ **Observation:** Generic variables don't occur in programs, merely model the structure of invariants; they also contribute linear to the polynomials

▽ Idea

Mark generic variables for special treatment in the reduction algorithm.
$\Longrightarrow$ Model of vectors and Modules

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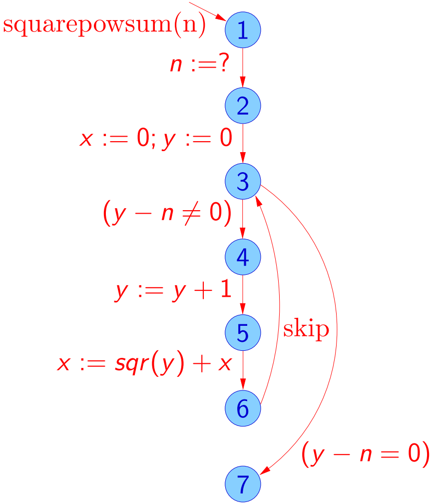
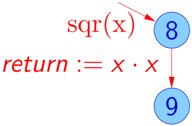
sqr (x ∈ ℕ) ∈ ℕ {
  return x · x;
}
```





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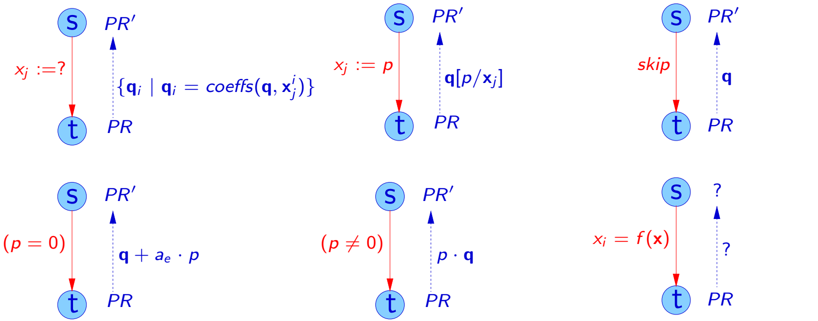
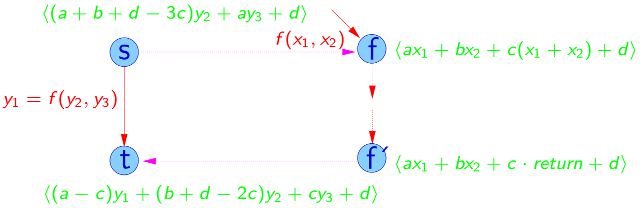


Method call details

Idea

Use precomputed templates to carry the effect of each method call.



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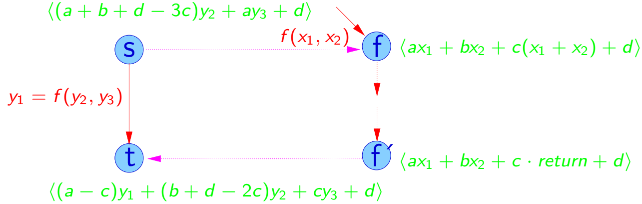

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Complete analysis

```
Set fixpointiteration (Node u_t, Vector v_t, Set Vars, Set Edges, Set Nodes) {
  Set [] G ⇐ new Set[|Nodes|];
  forall (u ∈ Nodes) G[u] ⇐ ∅;
  Set W ⇐ {(v_t, u_t)};
  while (W ≠ ∅) {
    (v, t) ⇐ extract(W);
    v ⇐ reduce(v, G[t]);
    if (v ≠ 0) {
      G[t] ⇐ G[t] ∪ {v};
      forall ( (s, ''skip;'' , t) ∈ Edges)
        W ⇐ W ∪ {(v, s)};
      forall ( (s, ''x_j := p'' , t) ∈ Edges)
        W ⇐ W ∪ {(v[p/x_j], s)};
      forall ( (s, ''(p ≠ 0)'' , t) ∈ Edges)
        W ⇐ W ∪ {(p · v, s)};
      forall ( (s, ''x_j :=?'' , t) ∈ Edges)
        let l = max({i|ax^i ∈ monoms(v)})
          in let v ⇒ (p_{0_0} x_j^0 + ... + p_{0_l} x_j^l, ..., p_{k_0} x_j^0 + ... + p_{k_l} x_j^l)
            in let v_i ⇐ (p_{0_i}, p_{1_i}, ..., p_{k_i})
              in  W ⇐ W ∪ {(v_0, u), ..., (v_l, u)};
    }
  }
  return ⟨G[u_start]⟩;
}
```

Benchmarks

Name	Calculation		ass-deg	Invariant	Time
geoSeries1	$x = (z-1) \cdot \sum_{k=0}^{K} z^k$	$y = z^K$	≤ 2	$x = y - 1$	$0,356s$
geoSeries2	$x = \sum_{k=0}^{K} z^k$	$y = z^{K-1}$	≤ 2	$x \cdot (z-1) = yz - 1$	$0,569s$
geoSeries3	$x = \sum_{k=0}^{K} a \cdot z^k$	$y = z^{K-1}$	≤ 2	$x \cdot (z-1) = azy - a$	$1,47s$

Name	Calculation		ass-deg	Invariant	Time
powSum1	$x = \sum_{k=0}^{K} 1$	$y = \sum_{k=0}^{K} 1$	≤ 1	$x = y$	$0,331s$
powSum2	$x = \sum_{k=0}^{K} k$	$y = \sum_{k=0}^{K} 1$	≤ 1	$2x = y^2 + y$	$0,776s$
powSum3	$x = \sum_{k=0}^{K} k^2$	$y = \sum_{k=0}^{K} 1$	≤ 2	$6x = 2y^3 + 3y^2 + y$	$1,47s$
powSum4	$x = \sum_{k=0}^{K} k^3$	$y = \sum_{k=0}^{K} 1$	≤ 3	$4x = y^4 + 2y^3 + y^2$	$2,71s$
powSum5	$x = \sum_{k=0}^{K} k^4$	$y = \sum_{k=0}^{K} 1$	≤ 4	$30x = 6y^5 + 15y^4 + 10y^3 - y$	$10,3s$
powSum6	$x = \sum_{k=0}^{K} k^5$	$y = \sum_{k=0}^{K} 1$	≤ 5	$12x = 2y^6 + 6y^5 + 5y^4 - y^2$	$787,2s$

Strategy	gs3/5	gs3/6	ps3/5	ps4/5	ps4/6	ps5/5	ps5/6	ps6/6
Original vector	8,4s	29,4s	3,83s	14,7s	...			
Reduced vector	7,3s	26,6s	2,9s	3,5s	8,1s	10,9s	30,0s	787s

Future Work

Implementation

- Treatment of procedure calls
- Scope on relevant variables
- Face large/real examples

Theory

Find a better upper complexity bound

$\Longrightarrow$ http://www2.cs.tum.edu/~petter/polyinvar

Thank You for Your attention!