Exercise Sheet 2

Exercise 1: (P/H) Lattices

Recall definitions of a lattice $\mathbb{D} = \langle D, \sqsubseteq_D \rangle$ and a complete lattice. Provide some examples of lattices and complete lattices. Let $\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}$ be lattices.

a) (P) Define a product lattice $\mathbb{D}_1 \times \mathbb{D}_2$. Show that $\mathbb{D}_1 \times \mathbb{D}_2$ is a complete lattice if $\mathbb{D}_1, \mathbb{D}_2$ are complete lattices.

b) (H) Define a lattice of functions $X \rightarrow \mathbb{D}$, where $X$ is a set. Show that $X \rightarrow \mathbb{D}$ is complete if $\mathbb{D}$ is complete.

c) (H) Define a lattice of monotonic functions $[\mathbb{D}_1 \rightarrow \mathbb{D}_2]$. Show that $[\mathbb{D}_1 \rightarrow \mathbb{D}_2]$ is complete if $\mathbb{D}_2$ is complete.

Exercise 2: (H) Multiple constraints for a single variable

Consider the following two constraint systems over a lattice $\mathbb{D}$:

\begin{align*}
(1) & \quad x \sqsubseteq f(x) \\
& \quad x \sqsubseteq g(x)
\end{align*}

\begin{align*}
(2) & \quad x \sqsubseteq f(x) \sqcup g(x)
\end{align*}

where $f, g: \mathbb{D} \rightarrow \mathbb{D}$. Show that the following hold:

a) $x$ is a solution of the constraint system (1) if and only if $x$ is a solution of the constraint system (2);

b) $x$ is the least solution of the constraint system (1) if and only if $x$ is the least solution of the constraint system (2).