Technische Universität München *Winter Semester 10/11*
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Program Optimization

Exercise Sheet 4

Deadline: November 22, at the lecture

Exercise 1: *(P) Monotonicity and Distributivity* *No Points*

Recall the lattice $\mathbb{D} = \mathbb{Z}_\bot^\top = \mathbb{Z} \cup \{\bot, \top\}$ with the ordering: $\forall x, y \in \mathbb{Z} : x \sqsubseteq y \iff x = y$, $\forall z \in \mathbb{D} : \bot \sqsubseteq z$, and $\forall z \in \mathbb{D} : z \sqsubseteq \top$. Determine for each of the following functions $\mathbb{D} \to \mathbb{D}$, whether it is monotonic, distributive, and strict.

$$bot(x) = \bot \qquad top(x) = \top$$

$$zero(x) = \begin{cases} \top & x = \top \\ 1 & x = 0 \\ 0 & \text{otherwise} \end{cases} \qquad inv(x) = \begin{cases} \top & x = \top \\ -x & x \in \mathbb{Z} \\ \bot & x = \bot \end{cases}$$

$$exp(x) = \begin{cases} \top & x = \top \\ 2^x & x \geq 0 \\ \bot & \text{otherwise} \end{cases} \qquad sq(x) = \begin{cases} \top & x = \top \\ x^2 & x \in \mathbb{Z} \\ \bot & x = \bot \end{cases}$$

Exercise 2: *Fixpoints versus constraints* *12 Points*

We wanted to solve a constraint system of the form $x \sqsupseteq F(x)$, where $F: \mathbb{D} \to \mathbb{D}$ is a monotonic function over a complete lattice $\mathbb{D}$ (e.g., it may be a vector-function representing all transfer functions from an analysis). Instead, we solve a fixpoint equation $x = F(x)$. The goal of this exercise is to justify this approach.

a) While it is obvious that a solution to $x = F(x)$ is a solution to $x \sqsupseteq F(x)$, give an example of a constraint system over some lattice and a solution to $x \sqsupseteq F(x)$ which is not a solution to $x = F(x)$.

b) If there are more solutions to the constraint system than to the fixpoint system, how do we know that the least fixpoint we compute is really the least solution to the constraint system? There may be better solutions to the constraint system that the fixpoint solving may not reach. Prove that this worry is unfounded: the least solution to $x = F(x)$ is also the least solution to $x \sqsupseteq F(x)$.

c) Also, show the reverse: the least solution of $x \sqsupseteq F(x)$ is the least solution of $x = F(x)$.

d) We solve the system $x = F(x)$ by fixpoint iteration from $\bot$. Assume we reach a fixpoint $F^i(\bot)$ after i iterations, why is this guaranteed to be the *least* fixpoint?

Exercise 3: *Chaotic iteration* *10 Points*

In the lecture, we solve the system through Round-Robin Iteration. Consider the following general iteration scheme, known as Chaotic Iteration. Using a similar notation as the lecture, for each iteration step d, we proceed as follows:

1. Pick any index i such that $x_i^{(d-1)} \sqsupseteq f_i(x^{(d-1)})$ is not satisfied.
2. If no such i exists, return $x^{(d-1)}$.
3. Otherwise, set $x_i^{(d)} = f_i(x^{(d-1)})$ and for all other indices $j \neq i$, set $x_j^{(d)} = x_j^{(d-1)}$. Increment d and start over.

It is obvious (from the exit condition) that the algorithm will not terminate until it has found a solution. Assuming that F is monotonic but the lattice may be of infinite height, determine if the following hold.

a) This chaotic scheme, if it terminates, returns the *least* fixpoint.

b) If the round-robin scheme of the lecture will terminate for a given constraint system, then the above scheme terminates for that system no matter how i is picked.