Exercise 1: \((P/H)\) Height of a lattice \hspace{1cm} 10 Points

For a lattice \(\mathbb{D}\), we define the height of the lattice \(h(\mathbb{D}) = n\) as the maximal length of strictly ascending chains \(d_0 \sqsubseteq d_1 \sqsubseteq d_2 \sqsubseteq \cdots \sqsubseteq d_n\) in the lattice. (Note that finiteness of height does not imply finiteness of a lattice. Give an example!)

Let \(\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}\) be lattices of finite height. Show that the following hold:

a) (P) \(h(\mathbb{D}_1 \times \mathbb{D}_2) = h(\mathbb{D}_1) + h(\mathbb{D}_2)\);

b) (H) \(h(\mathbb{D}^k) = k \cdot h(\mathbb{D})\), for any \(k \in \mathbb{N}\);

c) (H) \(h(X \rightarrow \mathbb{D}) = |X| \cdot h(\mathbb{D})\), where \(|X|\) is a cardinality of the finite set \(X\);

d) (H) \(h([\mathbb{D}_1 \rightarrow \mathbb{D}_2]) = |\mathbb{D}_1| \cdot h(\mathbb{D}_2)\), where \(\mathbb{D}_1\) is finite.

a) For, \(h(\mathbb{D}_1 \times \mathbb{D}_2) = h(\mathbb{D}_1) + h(\mathbb{D}_2)\), we abbreviate \(h, n\) and \(m\), for the heights of \(\mathbb{D}_1 \times \mathbb{D}_2, \mathbb{D}_1,\) and \(\mathbb{D}_2\), respectively. We want to prove that \(h = n + m\) by showing, first, \(h \geq n + m\) and, second, \(h \leq n + m\).

1. We know the sequence \(\bot \sqsubseteq a_1 \sqsubseteq \cdots \sqsubseteq a_n\) exists in \(\mathbb{D}_1\) and \(\bot \sqsubseteq b_1 \sqsubseteq \cdots \sqsubseteq b_m\) exists in \(\mathbb{D}_2\). We form the sequence \((\bot, \bot) \sqsubseteq (a_1, \bot) \sqsubseteq \cdots \sqsubseteq (a_n, \bot) \sqsubseteq (a_n, b_1) \sqsubseteq \cdots \sqsubseteq (a_n, b_m)\), which shows that \(h \geq n + m\).

2. Take an ascending chain \((a_0, b_0) \sqsubseteq \cdots \sqsubseteq (a_k, b_k)\). Now, a simple induction shows that \(|\{a_0, \ldots, a_k\}| + |\{b_0, \ldots, b_k\}| \geq k + 2\), because each strict inequality requires at least one element to differ. If we look at the elements \(A = \{a_0, \ldots, a_k\}\) and \(B = \{b_0, \ldots, b_k\}\), we see that these form ascending chains in \(\mathbb{D}_1\) and \(\mathbb{D}_2\), respectively. We obtain \(|A| \leq n + 1\) and \(|B| \leq m + 1\). (We need to add one, because a sequence of length \(k\) contains \(k + 1\) elements.) Putting it together, we have \(k + 2 \leq m + n + 2\), and \(h \leq m + n\).


c) You can either prove it directly, or reduce it to b) by showing that for the finite \(X\) there is an isomorphism between \(X \rightarrow \mathbb{D}\) and a product lattice.
A monotonic analysis framework is triple $⟨\mathcal{D}, \mathcal{F}, tf⟩$, where $\mathcal{D} = \langle 2^X, \subseteq \rangle$ is the complete lattice of subsets of a finite set $X$ (here, $\subseteq = \subseteq$), $\mathcal{F} \subseteq [\mathcal{D} \to \mathcal{D}]$ is a set of monotonic functions, and $tf : E \to \mathcal{F}$ is a mapping from edges in the control flow graph to transfer functions from $\mathcal{F}$. (This mapping $tf$ was denoted in the lecture by $[\_]^\ast$, i.e., $[k]^\ast \in \mathcal{F}$, for any $k \in E$.)

For a set $Y \subseteq X$, we denote its complement as $\overline{Y} = X \setminus Y$. Now, we define the complement of a monotonic analysis framework $⟨\mathcal{D}, \mathcal{F}, tf⟩$ as the triple $⟨\overline{\mathcal{D}}, \overline{\mathcal{F}}, \overline{tf}⟩$, where

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\begin{align*}
\overline{\mathcal{D}} &= \langle 2^X, \supseteq \rangle & \text{the lattice ordering is reversed, i.e., } \subseteq \overline{=} \supseteq, \\
\overline{f}(Y) &= f(\overline{Y}) & \text{complement of a function}, \\
\overline{\mathcal{F}} &= \{\overline{f} \mid f \in \mathcal{F}\} & \text{complement of a set of functions}, \\
\overline{tf}_{k} &= \overline{tf}_{k} & \text{complement of the transfer function, } k \in E.
\end{align*}
\]

a) Show that the complement of a monotonic analysis framework is itself a monotonic analysis framework.

b) Let $\mathcal{A}[u]$ (for all program points $u$) be a solution to the system

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\begin{align*}
& tf_{k}(\mathcal{A}[u]) \subseteq \mathcal{A}[v] \quad (k = (u, \_ , v) \in E).
\end{align*}
\]

Show that its complement $\overline{\mathcal{A}[u]}$ is a solution to the complementary analysis.

c) The complement of the Possibly Live Variables analysis described in the lecture is a Definitely Dead Variables analysis. Describe the lattice for this analysis and give the transfer function for assignments.