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Program Optimization

Solutions to Sheet 3

Exercise 1: *(P/H) Height of a lattice* — *10 Points*

For a lattice $\mathbb{D}$, we define the height of the lattice $h(\mathbb{D}) = n$ as the maximal length of strictly ascending chains $d_0 \sqsubset d_1 \sqsubset d_2 \sqsubset \cdots \sqsubset d_n$ in the lattice. (Note that finiteness of height does not imply finiteness of a lattice. Give an example!)

Let $\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}$ be lattices of finite height. Show that the following hold:

a) (P) $h(\mathbb{D}_1 \times \mathbb{D}_2) = h(\mathbb{D}_1) + h(\mathbb{D}_2)$;

b) (H) $h(\mathbb{D}^k) = k \cdot h(\mathbb{D})$, for any $k \in \mathbb{N}$;

c) (H) $h(X \to \mathbb{D}) = |X| \cdot h(\mathbb{D})$, where $|X|$ is a cardinality of the finite set X;

d) (H) $h([\mathbb{D}_1 \to \mathbb{D}_2]) = |\mathbb{D}_1| \cdot h(\mathbb{D}_2)$, where $\mathbb{D}_1$ is finite.

a) For, $h(\mathbb{D}_1 \times \mathbb{D}_2) = h(\mathbb{D}_1) + h(\mathbb{D}_2)$, we abbreviate h, n and m, for the heights of $\mathbb{D}_1 \times \mathbb{D}_2$, $\mathbb{D}_1$, and $\mathbb{D}_2$, respectively. We want to prove that $h = n + m$ by showing, first, $h \geq n + m$ and, second, $h \leq n + m$.

   1. We know the sequence $\bot \sqsubset a_1 \sqsubset \cdots \sqsubset a_n$ exists in $\mathbb{D}_1$ and $\bot \sqsubset b_1 \sqsubset \cdots \sqsubset b_m$ exists in $\mathbb{D}_2$. We form the sequence $(\bot, \bot) \sqsubset (a_1, \bot) \sqsubset \cdots \sqsubset (a_n, \bot) \sqsubset (a_n, b_1) \sqsubset \cdots \sqsubset (a_n, b_m)$, which shows that $h \geq n + m$.

   2. Take an ascending chain $(a_0, b_0) \sqsubset \cdots \sqsubset (a_k, b_k)$. Now, a simple induction shows that $|\{a_0, \ldots, a_k\}| + |\{b_0, \ldots, b_k\}| \geq k + 2$, because each strict inequality requires at least one element to differ. If we look at the elements $A = \{a_0, \ldots, a_k\}$ and $B = \{b_0, \ldots, b_k\}$, we see that these form ascending chains in $\mathbb{D}_1$ and $\mathbb{D}_2$, respectively. We obtain $|A| \leq n + 1$ and $|B| \leq m + 1$. (We need to add one, because a sequence of length k contains $k + 1$ elements.) Putting it together, we have $k + 2 \leq m + n + 2$, and $h \leq m + n$.

b) Use induction, Luke.

c) You can either prove it directly, or reduce it to b) by showing that for the finite X there is an isomorphism between $X \to \mathbb{D}$ and a product lattice.

Exercise 2: *(H) Monotonic Analysis Frameworks* *10 Points*

A monotonic analysis framework is triple $\langle \mathbb{D}, \mathcal{F}, tf \rangle$, where $\mathbb{D} = \langle 2^X, \sqsubseteq \rangle$ is the complete lattice of subsets of a finite set X (here, $\sqsubseteq = \subseteq$), $\mathcal{F} \subseteq [\mathbb{D} \to \mathbb{D}]$ is a set of monotonic functions, and $tf : E \to \mathcal{F}$ is a mapping from edges in the control flow graph to transfer functions from $\mathcal{F}$. (This mapping tf was denoted in the lecture by $[\![-]\!]^\sharp$, i.e., $[\![k]\!]^\sharp \in \mathcal{F}$, for any $k \in E$.)

For a set $Y \subseteq X$, we denote its complement as $\overline{Y} = X \setminus Y$. Now, we define the complement of a monotonic analysis framework $\langle \mathbb{D}, \mathcal{F}, tf \rangle$ as the triple $\langle \overline{\mathbb{D}}, \overline{\mathcal{F}}, \overline{tf} \rangle$, where

$$\overline{\mathbb{D}} = \langle 2^X, \sqsupseteq \rangle \qquad \text{the lattice ordering is } \textit{reversed}\text{, i.e., } \sqsubseteq_{\overline{\mathbb{D}}} = \supseteq,$$

$$\overline{f}(Y) = \overline{f(\overline{Y})} \qquad \text{complement of a function,}$$

$$\overline{\mathcal{F}} = \{\overline{f} \mid f \in \mathcal{F}\} \qquad \text{complement of a set of functions,}$$

$$\overline{tf}_k = \overline{tf_k} \qquad \text{complement of the transfer function, } k \in E.$$

a) Show that the complement of a monotonic analysis framework is itself a monotonic analysis framework.

b) Let $\mathcal{A}[u]$ (for all program points u) be a solution to the system

$$tf_k(\mathcal{A}[u]) \sqsubseteq \mathcal{A}[v] \quad (k = (u, \_, v) \in E)\,.$$

Show that its complement $\overline{A[u]}$ is a solution to the complementary analysis.

c) The complement of the Possibly Live Variables analysis described in the lecture is a Definitely Dead Variables analysis. Describe the lattice for this analysis and give the transfer function for assignments.