Example:

$$5y - 10z = 18$$

has no solution over $\mathbb{Z}$:-)

Observation 2:

Adding a multiple of one equation to another does not change the set of solutions :-)

Example:

$$\begin{array}{rcrcrcr} 2x & + & 3y & & & = & 24 \\ x & - & y & + & 5z & = & 3 \end{array}$$

Example:

$$\begin{array}{rcrcrcr} 2x & + & 3y & & & = & 24 \\ x & - & y & + & 5z & = & 3 \end{array}$$

$$\Longrightarrow$$

$$\begin{array}{rcrcrcr} & & 5y & - & 10z & = & 18 \\ x & - & y & + & 5z & = & 3 \end{array}$$

Observation 3:

Adding multiples of columns to another column is an invertible transformation which we keep track of in a separate matrix ...

$$\begin{array}{ccc|ccccccc} 1 & 0 & 0 & & & 5y & - & 10z & = & 18 \\ 0 & 1 & 0 & x & - & y & + & 5z & = & 3 \\ 0 & 0 & 1 & & & & & & & \end{array}$$

$$\Longrightarrow$$

$$\begin{array}{ccc|ccccccc} 1 & 0 & 0 & & & 5y & & & = & 18 \\ 0 & 1 & 2 & x & - & y & + & 3z & = & 3 \\ 0 & 0 & 1 & & & & & & & \end{array}$$

Observation 3:

Adding multiples of columns to another column is an invertible transformation which we keep track of in a separate matrix ...

$$\begin{array}{ccc|ccccccc} 1 & 0 & 0 & & & 5y & & & = & 18 \\ 0 & 1 & 2 & x & - & y & + & 3z & = & 3 \\ 0 & 0 & 1 & & & & & & & \end{array}$$

$$\Longrightarrow$$

$$\begin{array}{ccc|ccccccc} 1 & 0 & -3 & & & 5y & & & = & 18 \\ 0 & 1 & 2 & x & - & y & & & = & 3 \\ 0 & 0 & 1 & & & & & & & \end{array}$$

$\Longrightarrow$ triangular form !!

Observation 4:

- A special solution of a triangular system can be directly read off :-)
- All solutions of a homogeneous triangular system can be directly read off :-)
- All solutions of the original system can be recovered from the solutions of the triangular system by means of the accumulated transformation matrix:-))

Example

$$\begin{array}{ccc|ccccc} 1 & 0 & -3 & & & 5y & = & 15 \\ 0 & 1 & 2 & x & - & y & = & 3 \\ 0 & 0 & 1 & & & & & \end{array}$$

One special solution:

$$[6, 3, 0]^\top$$

All solutions of the homogeneous system are spanned by:

$$[0, 0, 1]^\top$$

Solving over $\mathbb{N}$

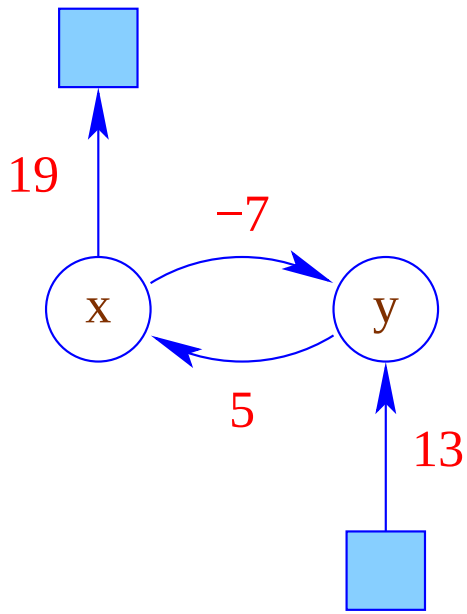
- ... is of major practical importance;
- ... has led to the development of many new techniques;
- ... easily allows to encode NP-hard problems;
- ... remains difficult if just three variables are allowed per equation.

2. One Polynomial Special Case:

$$\begin{array}{rcrcl} & & x & \geq & y+5 \\ 19 & \geq & x & & \\ & & y & \geq & 13 \\ & & y & \geq & x-7 \end{array}$$

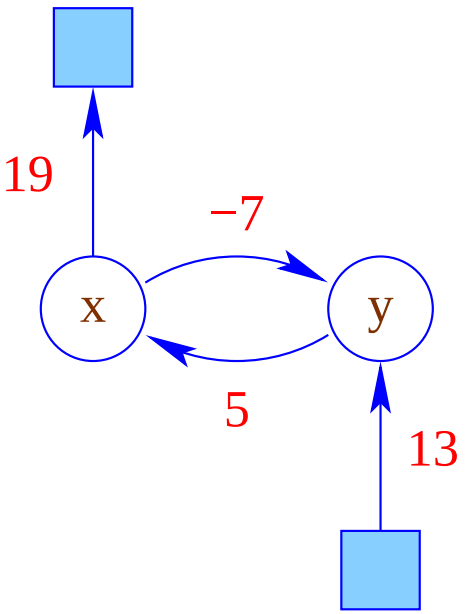
- There are at most 2 variables per in-equation;
- no scaling factors.

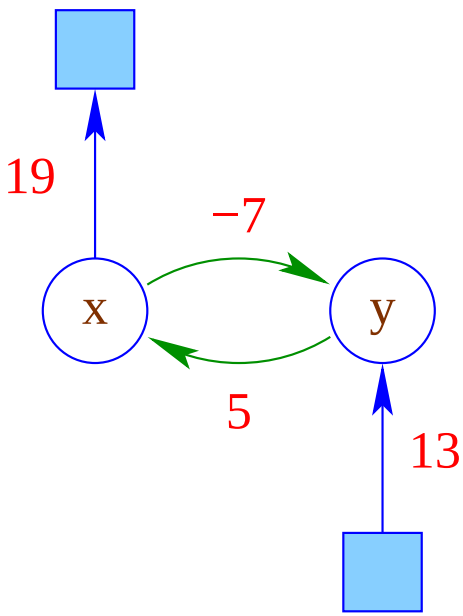
Idea: Represent the system by a graph:

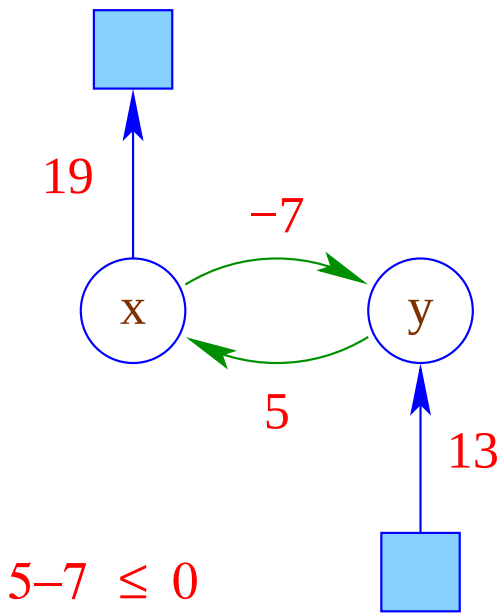


The in-equations are satisfiable iff

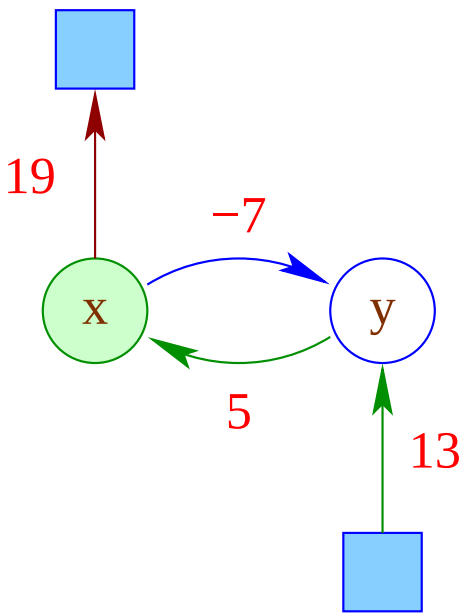
- the weight of every cycle are at most 0;
- the weights of paths reaching x are bounded by the weights of edges from x into the sink.

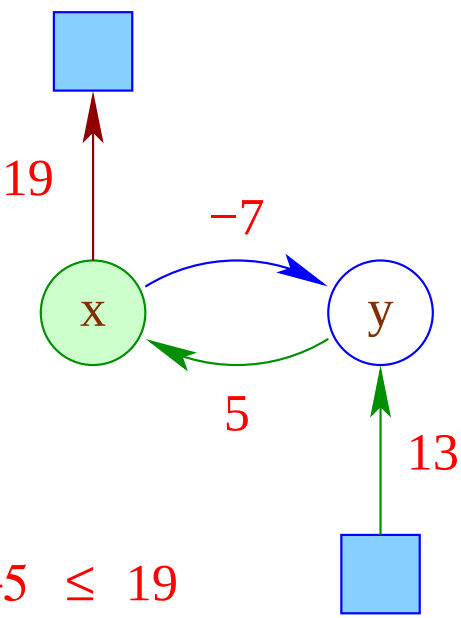

19
−7
x
y
5
13


19
−7
x
y
5
13



$5-7 \leq 0$


19
−7
x
y
5
13



$13+5 \leq 19$

The in-equations are satisfiable iff

- the weight of every cycle are at most 0;
- the weights of paths reaching x are bounded by the weights of edges from x into the sink.

$$\Longrightarrow$$

Compute the reflexive and transitive closure of the edge weights!

3. A General Solution Method:

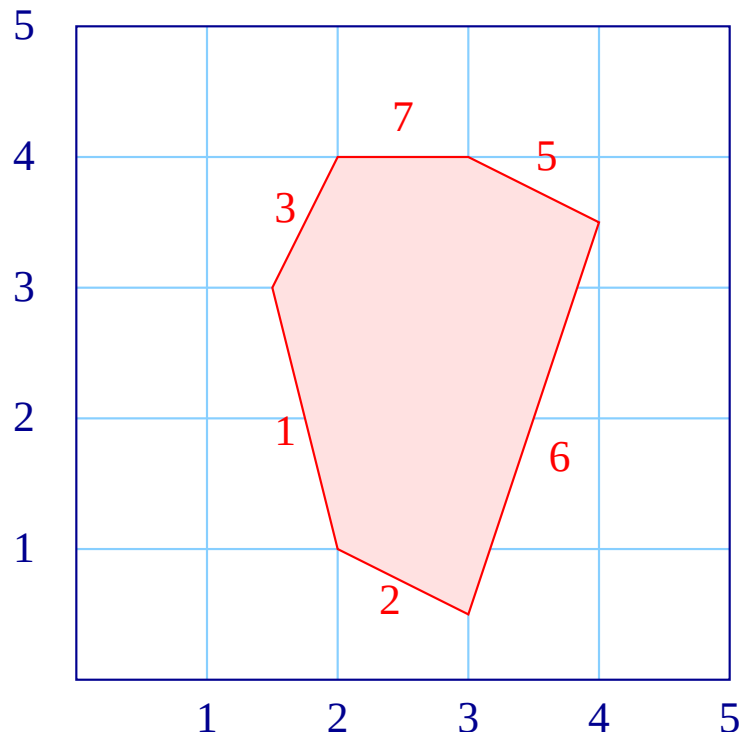
Idea: Fourier-Motzkin Elimination

- Successively remove individual variables x !
- All in-equations with positive occurrences of x yield lower bounds.
- All in-equations with negative occurrences of x yield upper bounds.
- All lower bounds must be at most as big as all upper bounds ;-))

Jean Baptiste Joseph Fourier, 1768–1830

Example:

$$
\begin{aligned}
9 &\leq 4x_1 + x_2 && (1)\\
4 &\leq x_1 + 2x_2 && (2)\\
0 &\leq 2x_1 - x_2 && (3)\\
6 &\leq x_1 + 6x_2 && (4)\\
-11 &\leq -x_1 - 2x_2 && (5)\\
-17 &\leq -6x_1 + 2x_2 && (6)\\
-4 &\leq -x_2 && (7)
\end{aligned}
$$



For x_1 we obtain:

$$
\begin{array}{lcll}
9 & \leq & 4x_1 + x_2 & (1) \\
4 & \leq & x_1 + 2x_2 & (2) \\
0 & \leq & 2x_1 - x_2 & (3) \\
6 & \leq & x_1 + 6x_2 & (4) \\
-11 & \leq & -x_1 - 2x_2 & (5) \\
-17 & \leq & -6x_1 + 2x_2 & (6) \\
-4 & \leq & -x_2 & (7)
\end{array}
\qquad
\begin{array}{lcll}
\frac{9}{4} - \frac{1}{4}x_2 & \leq & x_1 & (1) \\
4 - 2x_2 & \leq & x_1 & (2) \\
\frac{1}{2}x_2 & \leq & x_1 & (3) \\
6 - 6x_2 & \leq & x_1 & (4) \\
x_1 & \leq & 11 - 2x_2 & (5) \\
x_1 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (6) \\
-4 & \leq & -x_2 & (7)
\end{array}
$$

If such an x_1 exists, all lower bounds must be bounded by all upper bounds, i.e.,

$$
\begin{array}{rcl@{\qquad}c}
\frac{9}{4} - \frac{1}{4}x_2 & \leq & 11 - 2x_2 & (1,5) \\
\frac{9}{4} - \frac{1}{4}x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (1,6) \\
4 - 2x_2 & \leq & 11 - 2x_2 & (2,5) \\
4 - 2x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (2,6) \\
\frac{1}{2}x_2 & \leq & 11 - 2x_2 & (3,5) \\
\frac{1}{2}x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (3,6) \\
6 - 6x_2 & \leq & 11 - 2x_2 & (4,5) \\
6 - 6x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (4,6) \\
-4 & \leq & -x_2 & (7)
\end{array}
\quad \text{or} \quad
\begin{array}{rcl@{\qquad}c}
-35 & \leq & -7x_2 & (1,5) \\
-\frac{7}{12} & \leq & \frac{7}{12}x_2 & (1,6) \\
-7 & \leq & 0 & (2,5) \\
\frac{7}{6} & \leq & \frac{7}{3}x_2 & (2,6) \\
-22 & \leq & -5x_2 & (3,5) \\
-\frac{17}{6} & \leq & -\frac{1}{6}x_2 & (3,6) \\
-5 & \leq & 4x_2 & (4,5) \\
\frac{19}{6} & \leq & \frac{19}{3}x_2 & (4,6) \\
-4 & \leq & -x_2 & (7)
\end{array}
$$

$$
\begin{array}{rcll}
\frac{9}{4} - \frac{1}{4}x_2 & \leq & 11 - 2x_2 & (1,5) \\
\frac{9}{4} - \frac{1}{4}x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (1,6) \\
4 - 2x_2 & \leq & 11 - 2x_2 & (2,5) \\
4 - 2x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (2,6) \\
\frac{1}{2}x_2 & \leq & 11 - 2x_2 & (3,5) \\
\frac{1}{2}x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (3,6) \\
6 - 6x_2 & \leq & 11 - 2x_2 & (4,5) \\
6 - 6x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (4,6) \\
-4 & \leq & -x_2 & (7)
\end{array}
\quad \text{or} \quad
\begin{array}{rcll}
-5 & \leq & -x_2 & (1,5) \\
-1 & \leq & x_2 & (1,6) \\
-7 & \leq & 0 & (2,5) \\
\frac{1}{2} & \leq & x_2 & (2,6) \\
-\frac{22}{5} & \leq & -x_2 & (3,5) \\
-17 & \leq & -x_2 & (3,6) \\
-\frac{5}{4} & \leq & x_2 & (4,5) \\
\frac{1}{2} & \leq & x_2 & (4,6) \\
-4 & \leq & -x_2 & (7)
\end{array}
$$

This is the one-variable case which we can solve exactly:

$$\max\{-1, \boxed{\tfrac{1}{2}}, -\tfrac{5}{4}, \tfrac{1}{2}\} \quad \leq \quad x_2 \quad \leq \quad \min\{5, \tfrac{22}{5}, 17, \boxed{4}\}$$

From which we conclude: $x_2 \in [\frac{1}{2}, 4]$:-)

In General:

- The original system has a solution over $\mathbb{Q}$ iff the system after elimination of one variable has a solution over $\mathbb{Q}$:-)
- Every elimination step may square the number of in-equations $\Longrightarrow$ exponential run-time :-((
- It can be modified such that it also decides satisfiability over $\mathbb{Z}$ $\Longrightarrow$ Omega Test

William Worthington Pugh, Jr.
University of Maryland, College Park

Idea:

- We successively remove variables. Thereby we omit division ...
- If x only occurs with coefficient ± 1, we apply Fourier-Motzkin elimination :-)
- Otherwise, we provide a bound for a positive multiple of x ...

Consider, e.g., (1) and (6) :

$$\begin{aligned} 6 \cdot x_1 &\leq 17 + 2x_2 \\ 9 - x_2 &\leq 4 \cdot x_1 \end{aligned}$$

W.l.o.g., we only consider strict in-equations:

$$\begin{aligned} 6 \cdot x_1 &< 18 + 2x_2 \\ 8 - x_2 &< 4 \cdot x_1 \end{aligned}$$

... where we always divide by gcds:

$$\begin{aligned} 3 \cdot x_1 &< 9 + x_2 \\ 8 - x_2 &< 4 \cdot x_1 \end{aligned}$$

This implies:

$$3 \cdot (8 - x_2) < 4 \cdot (9 + x_2)$$

We thereby obtain:

- If one derived in-equation is unsatisfiable, then also the overall system :-)
- If all derived in-equations are satisfiable, then there is a solution which, however, need not be integer :-(
- An integer solution is guaranteed to exist if there is sufficient separation between lower and upper bound ...
- Assume $\alpha < a \cdot x$ $\qquad$ $b \cdot x < \beta$.

  Then it should hold that:

$$b \cdot \alpha < a \cdot \beta$$

  and moreover:

$$\boxed{a \cdot b} < a \cdot \beta - b \cdot \alpha$$

... in the Example:

$$12 \quad < \quad 4 \cdot (9 + x_2) - 3 \cdot (8 - x_2)$$

or:

$$12 \quad < \quad 12 + 7x_2$$

or:

$$0 \quad < \quad x_2$$

In the example, also these strengthened in-equations are satisfiable

$\Longrightarrow$ the system has a solution over $\mathbb{Z}$:-)