Example:

\[ 5y - 10z = 18 \]

has no solution over \( \mathbb{Z} \)  :-)

Observation 2:

Adding a multiple of one equation to another does not change the set of solutions  :-)}
Example:

\[ 2x + 3y = 24 \]
\[ x - y + 5z = 3 \]
Example:

\[
\begin{align*}
2x & \quad + \quad 3y & \quad = \quad 24 \\
    x & \quad - \quad y & \quad + \quad 5z & \quad = \quad 3
\end{align*}
\]

\[\Rightarrow\]

\[
\begin{align*}
5y & \quad - \quad 10z & \quad = \quad 18 \\
    x & \quad - \quad y & \quad + \quad 5z & \quad = \quad 3
\end{align*}
\]
Observation 3:

Adding multiples of columns to another column is an invertible transformation which we keep track of in a separate matrix ...

\[
\begin{bmatrix}
1 & 0 & 0 & \quad 5y - 10z = 18 \\
0 & 1 & 0 & \quad x - y + 5z = 3 \\
0 & 0 & 1 \\
\end{bmatrix} \quad \Rightarrow \quad \\
\begin{bmatrix}
1 & 0 & 0 & \quad 5y = 18 \\
0 & 1 & 2 & \quad x - y + 3z = 3 \\
0 & 0 & 1 \\
\end{bmatrix}
\]
Observation 3:

Adding multiples of columns to another column is an invertible transformation which we keep track of in a separate matrix ...

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{bmatrix}
\begin{align*}
5y &= 18 \\
x - y + 3z &= 3
\end{align*}
\Rightarrow
\begin{bmatrix}
1 & 0 & -3 \\
0 & 1 & 2 \\
0 & 0 & 1
\end{bmatrix}
\begin{align*}
5y &= 18 \\
x - y &= 3
\end{align*}
\Rightarrow \text{triangular form}!!
Observation 4:

- A special solution of a triangular system can be directly read off :-)

- All solutions of a homogeneous triangular system can be directly read off :-)

- All solutions of the original system can be recovered from the solutions of the triangular system by means of the accumulated transformation matrix:-))
Example

\[
\begin{array}{ccc|c}
1 & 0 & -3 & 5y \\
0 & 1 & 2 & x - y \\
0 & 0 & 1 &
\end{array}
\]

One special solution:

\[ [6, 3, 0]^\top \]

All solutions of the homogeneous system are spanned by:

\[ [0, 0, 1]^\top \]
Solving over \( \mathbb{N} \)

- ... is of major practical importance;

- ... has led to the development of many new techniques;

- ... easily allows to encode \textit{NP-hard} problems;

- ... remains difficult if just \textit{three} variables are allowed per equation.
2. One Polynomial Special Case:

\[ x \geq y + 5 \]

\[ 19 \geq x \]

\[ y \geq 13 \]

\[ y \geq x - 7 \]

- There are at most 2 variables per in-equation;
- no scaling factors.
Idea: Represent the system by a graph:
The in-equations are *satisfiable* iff

- the weight of every *cycle* are at most $0$;
- the weights of paths *reaching* $x$ are bounded by the weights of edges from $x$ into the *sink*. 
\[ 5 - 7 \leq 0 \]
13 + 5 ≤ 19
The in-equations are **satisfiable** iff

- the weight of every **cycle** are at most \( 0 \);
- the weights of paths **reaching** \( x \) are bounded by the weights of edges from \( x \) into the **sink**.

\[ \Rightarrow \]

Compute the **reflexive** and **transitive** closure of the edge weights!
3. **A General Solution Method:**

**Idea:** Fourier-Motzkin Elimination

- Successively remove individual variables $x$!
- All in-equations with **positive** occurrences of $x$ yield **lower bounds**.
- All in-equations with **negative** occurrences of $x$ yield **upper bounds**.
- All lower bounds must be at most as big as all upper bounds ;-)
Jean Baptiste Joseph Fourier, 1768–1830
Example:

\[
9 \leq 4x_1 + x_2 \quad (1)
\]

\[
4 \leq x_1 + 2x_2 \quad (2)
\]

\[
0 \leq 2x_1 - x_2 \quad (3)
\]

\[
6 \leq x_1 + 6x_2 \quad (4)
\]

\[
-11 \leq -x_1 - 2x_2 \quad (5)
\]

\[
-17 \leq -6x_1 + 2x_2 \quad (6)
\]

\[
-4 \leq -x_2 \quad (7)
\]
For \( x_1 \) we obtain:

\[
\begin{align*}
9 & \leq 4x_1 + x_2 \quad (1) \\
4 & \leq x_1 + 2x_2 \quad (2) \\
0 & \leq 2x_1 - x_2 \quad (3) \\
6 & \leq x_1 + 6x_2 \quad (4) \\
-11 & \leq -x_1 - 2x_2 \quad (5) \\
-17 & \leq -6x_1 + 2x_2 \quad (6) \\
-4 & \leq -x_2 \quad (7)
\end{align*}
\]

\[
\begin{align*}
\frac{9}{4} - \frac{1}{4}x_2 & \leq x_1 \quad (1) \\
4 - 2x_2 & \leq x_1 \quad (2) \\
\frac{1}{2}x_2 & \leq x_1 \quad (3) \\
6 - 6x_2 & \leq x_1 \quad (4) \\
x_1 & \leq 11 - 2x_2 \quad (5) \\
x_1 & \leq \frac{17}{6} + \frac{1}{3}x_2 \quad (6) \\
-4 & \leq -x_2 \quad (7)
\end{align*}
\]

If such an \( x_1 \) exists, all lower bounds must be bounded by all upper bounds, i.e.,
\[
\frac{9}{4} - \frac{1}{4}x_2 \leq 11 - 2x_2 \quad (1, 5) \\
-35 \leq -7x_2 \quad (1, 5)
\]

\[
\frac{9}{4} - \frac{1}{4}x_2 \leq \frac{17}{6} + \frac{1}{3}x_2 \quad (1, 6) \\
-\frac{7}{12} \leq \frac{7}{12}x_2 \quad (1, 6)
\]

\[
4 - 2x_2 \leq 11 - 2x_2 \quad (2, 5) \\
-7 \leq 0 \quad (2, 5)
\]

\[
4 - 2x_2 \leq \frac{17}{6} + \frac{1}{3}x_2 \quad (2, 6) \\
\frac{7}{6} \leq \frac{7}{3}x_2 \quad (2, 6)
\]

\[
\frac{1}{2}x_2 \leq 11 - 2x_2 \quad (3, 5) \\
-22 \leq -5x_2 \quad (3, 5)
\]

\[
\frac{1}{2}x_2 \leq \frac{17}{6} + \frac{1}{3}x_2 \quad (3, 6) \\
-\frac{17}{6} \leq -\frac{1}{6}x_2 \quad (3, 6)
\]

\[
6 - 6x_2 \leq 11 - 2x_2 \quad (4, 5) \\
-5 \leq 4x_2 \quad (4, 5)
\]

\[
6 - 6x_2 \leq \frac{17}{6} + \frac{1}{3}x_2 \quad (4, 6) \\
\frac{19}{6} \leq \frac{19}{3}x_2 \quad (4, 6)
\]

\[
-4 \leq -x_2 \quad (7) \\
-4 \leq -x_2 \quad (7)
\]

This is the one-variable case which we can solve exactly:
\[
\begin{align*}
\frac{9}{4} - \frac{1}{4}x_2 & \leq 11 - 2x_2 \quad (1, 5) \\
\frac{9}{4} - \frac{1}{4}x_2 & \leq \frac{17}{6} + \frac{1}{3}x_2 \quad (1, 6) \\
4 - 2x_2 & \leq 11 - 2x_2 \quad (2, 5) \\
4 - 2x_2 & \leq \frac{17}{6} + \frac{1}{3}x_2 \quad (2, 6) \\
\frac{1}{2}x_2 & \leq 11 - 2x_2 \quad (3, 5) \quad \text{or} \quad -\frac{22}{5} \leq -x_2 \quad (3, 5) \\
\frac{1}{2}x_2 & \leq \frac{17}{6} + \frac{1}{3}x_2 \quad (3, 6) \\
6 - 6x_2 & \leq 11 - 2x_2 \quad (4, 5) \\
6 - 6x_2 & \leq \frac{17}{6} + \frac{1}{3}x_2 \quad (4, 6) \\
-4 & \leq -x_2 \quad (7) \\
\end{align*}
\]

This is the one-variable case which we can solve exactly:
\[
\max \left\{-1, \frac{1}{2}, -\frac{5}{4}, \frac{1}{2}\right\} \leq x_2 \leq \min \left\{5, \frac{22}{5}, 17, 4\right\}
\]

From which we conclude: \(x_2 \in \left[\frac{1}{2}, 4\right]\) :-)

**In General:**

- The original system has a solution over \(\mathbb{Q}\) iff the system after elimination of one variable has a solution over \(\mathbb{Q}\) :-)
- Every elimination step may **square** the number of in-equations \(\Longrightarrow\) exponential run-time :-((
- It can be modified such that it also decides satisfiability over \(\mathbb{Z}\) \(\Longrightarrow\) Omega Test
William Worthington Pugh, Jr.
University of Maryland, College Park
Idea:

- We successively remove variables. Thereby we omit division ...
- If $x$ only occurs with coefficient $\pm 1$, we apply Fourier-Motzkin elimination :-)
- Otherwise, we provide a bound for a positive multiple of $x$ ...

Consider, e.g., (1) and (6):

\[
\begin{align*}
6 \cdot x_1 & \leq 17 + 2x_2 \\
9 - x_2 & \leq 4 \cdot x_1
\end{align*}
\]
W.l.o.g., we only consider strict in-equations:

\[ 6 \cdot x_1 \ < \ 18 + 2 x_2 \]
\[ 8 - x_2 \ < \ 4 \cdot x_1 \]

... where we always divide by gcds:

\[ 3 \cdot x_1 \ < \ 9 + x_2 \]
\[ 8 - x_2 \ < \ 4 \cdot x_1 \]

This implies:

\[ 3 \cdot (8 - x_2) \ < \ 4 \cdot (9 + x_2) \]
We thereby obtain:

- If one derived in-equation is *unsatisfiable*, then also the overall system :-)  
- If all derived in-equations are satisfiable, then there is a solution which, however, need not be *integer* :-(  
- An integer solution is guaranteed to exist if there is *sufficient separation* between lower and upper bound ...  
- Assume \( \alpha < a \cdot x \) \quad \text{and} \quad b \cdot x < \beta . \)

Then it should hold that:

\[ b \cdot \alpha < a \cdot \beta \]

and moreover:

\[ a \cdot b < a \cdot \beta - b \cdot \alpha \]
... in the Example:

\[ 12 < 4 \cdot (9 + x_2) - 3 \cdot (8 - x_2) \]

or:

\[ 12 < 12 + 7x_2 \]

or:

\[ 0 < x_2 \]

In the example, also these strengthened in-equations are satisfiable

\[ \implies \text{the system has a solution over } \mathbb{Z} \quad :-) \]