$$
\begin{array}{lcl}
[\![x = M[e];]\!]^\sharp\, A & = & (A \cup \{e\}) \backslash Expr_x \\
[\![M[e_1] = e_2;]\!]^\sharp\, A & = & A \cup \{e_1, e_2\}
\end{array}
$$

By that, every path can be analyzed :-)

A given program may admit several paths :-(

For any given input, another path may be chosen :-((

$\Longrightarrow$ We require the set:

$$
\mathcal{A}[v] \quad = \quad \bigcap \{ [\![\pi]\!]^\sharp \emptyset \mid \pi : start \rightarrow^* v \}
$$

Concretely:

- $\rightarrow$ We consider all paths π which reach v.
- $\rightarrow$ For every path π, we determine the set of expressions which are available along π.
- $\rightarrow$ Initially at program start, nothing is available :-)
- $\rightarrow$ We compute the intersection $\Longrightarrow$ safe information

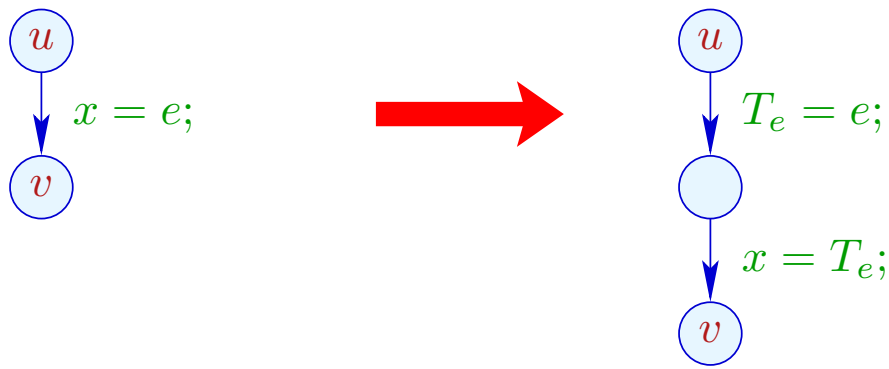
Concretely:

- $\rightarrow$ We consider all paths π which reach v.
- $\rightarrow$ For every path π, we determine the set of expressions which are available along π.
- $\rightarrow$ Initially at program start, nothing is available :-)
- $\rightarrow$ We compute the intersection $\Longrightarrow$ safe information

How do we exploit this information ???

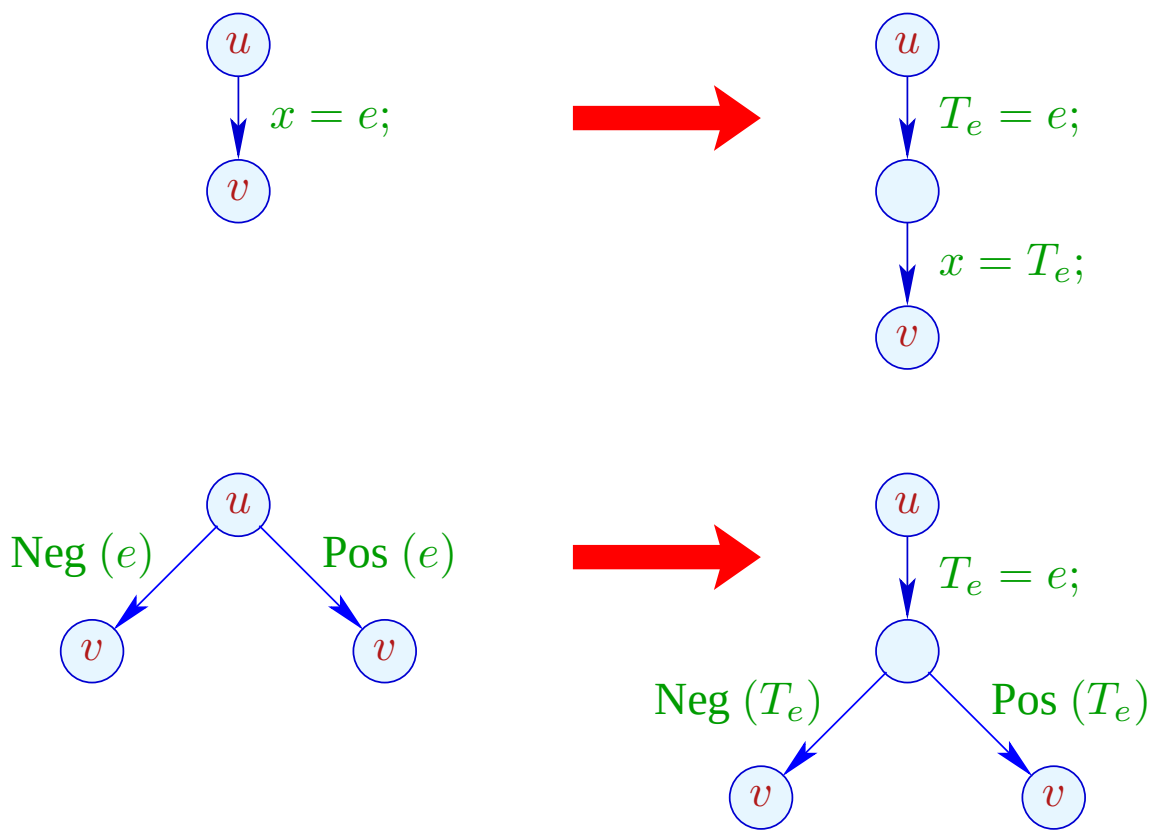
Transformation 1.1:

We provide novel registers T_e as storage for the e:



Transformation 1.1:

We provide novel registers T_e as storage for the e:



... analogously for $R = M[e];$ and $M[e_1] = e_2;$.

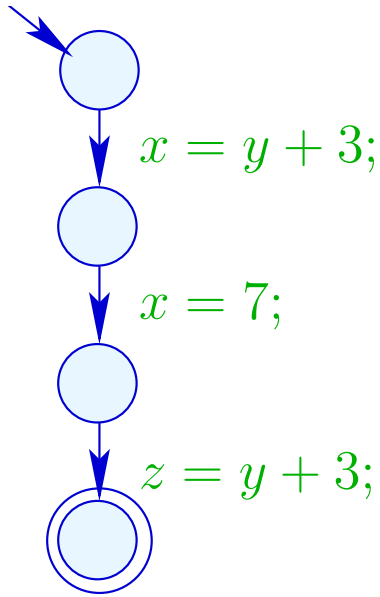
Transformation 1.2:

If e is available at program point u, then e need not be re-evaluated:

We replace the assignment with Nop :-)

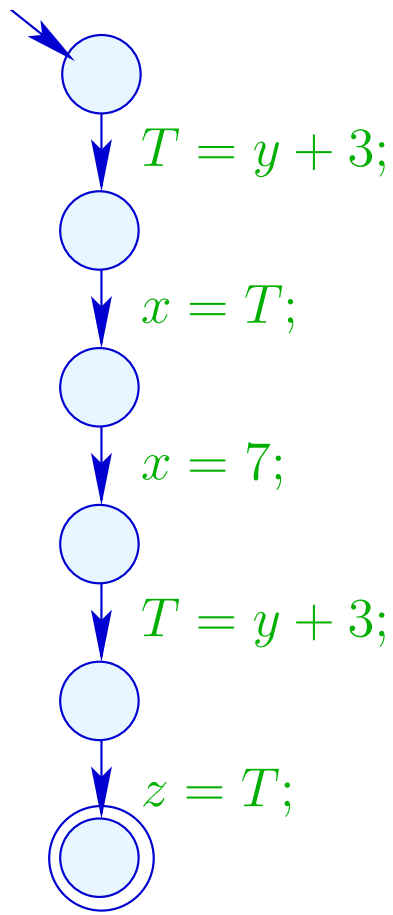
Example:

$$
\begin{aligned}
x &= y + 3; \\
x &= 7; \\
z &= y + 3;
\end{aligned}
$$



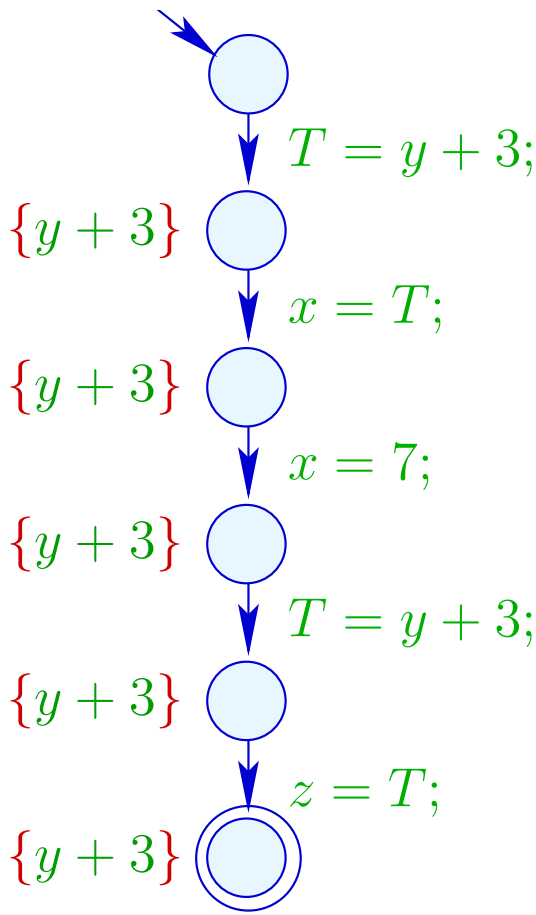
Example:

$$
\begin{aligned}
x &= y + 3; \\
x &= 7; \\
z &= y + 3;
\end{aligned}
$$



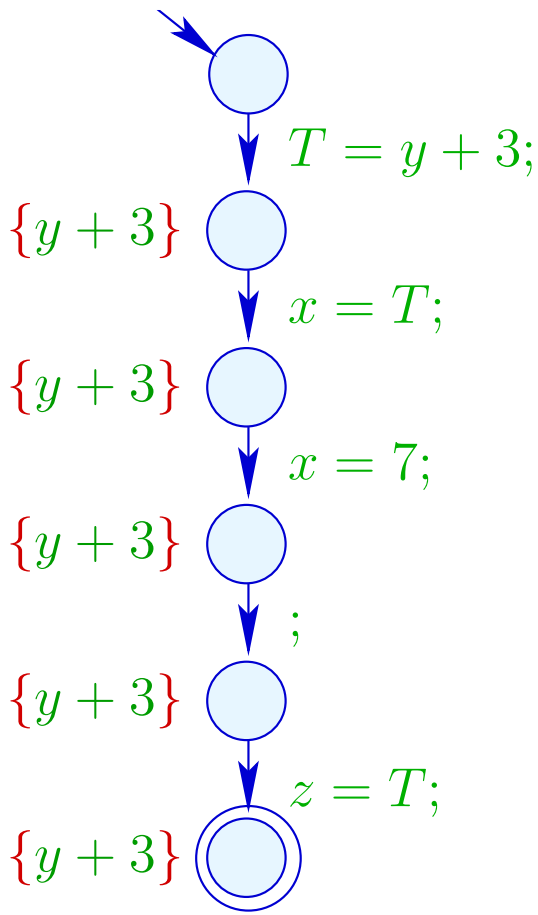
Example:

$$
\begin{aligned}
x &= y + 3; \\
x &= 7; \\
z &= y + 3;
\end{aligned}
$$



Example:

$$
\begin{aligned}
x &= y + 3; \\
x &= 7; \\
z &= y + 3;
\end{aligned}
$$



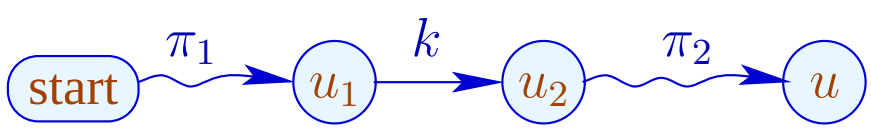
Correctness: (Idea)

Transformation 1.1 preserves the semantics and $\mathcal{A}[u]$ for all program points u :-)

Assume $\pi : start \rightarrow^* u$ is the path taken by a computation.

If $e \in \mathcal{A}[u]$, then also $e \in [\![\pi]\!]^\sharp\, \emptyset$.

Therefore, π can be decomposed into:



with the following properties:

- The expression e is evaluated at the edge k;
- The expression e is not removed from the set of available expressions at any edge in π_2, i.e., no variable of e receives a new value :-)

- The expression e is evaluated at the edge k;
- The expression e is not removed from the set of available expressions at any edge in π_2, i.e., no variable of e receives a new value :-)

$\Longrightarrow$

The register T_e contains the value of e whenever u is reached :-))

Warning:

Transformation 1.1 is only meaningful for assignments $x = e;$ where:

$\rightarrow$ $e \notin Vars;$

$\rightarrow$ the evaluation of e is non-trivial :-}

Warning:

Transformation 1.1 is only meaningful for assignments $x = e;$ where:

$\rightarrow$ $x \notin Vars(e)$;

$\rightarrow$ $e \notin Vars$;

$\rightarrow$ the evaluation of e is non-trivial :- }

Which leaves us with the following question ...

Question:

How do we compute $\mathcal{A}[u]$ for every program point u ??

Question:

How can we compute $\mathcal{A}[u]$ for every program point u ??

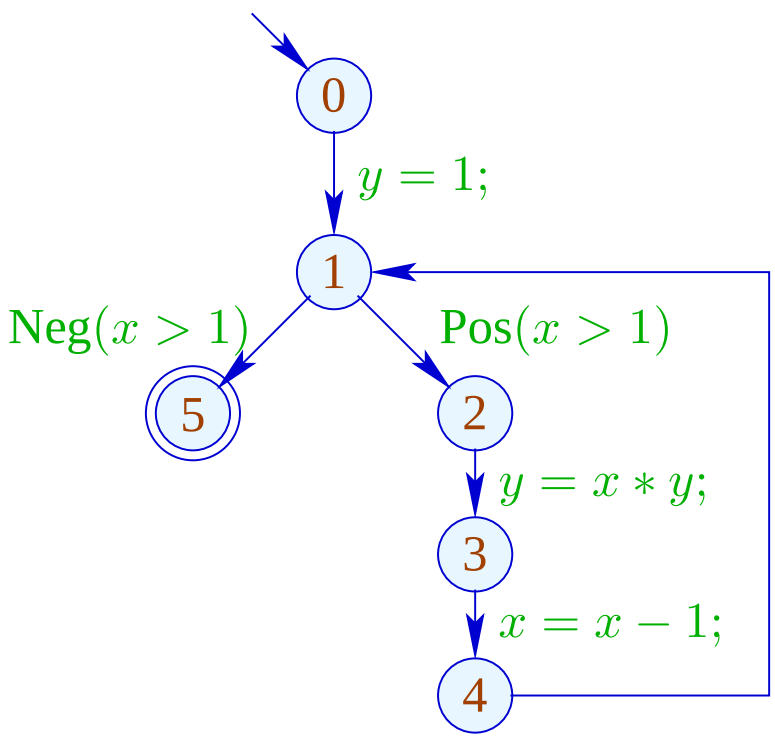
We collect all restrictions to the values of $\mathcal{A}[u]$ into a system of constraints:

$$
\begin{array}{lll}
\mathcal{A}[start] & \subseteq & \emptyset \\
\mathcal{A}[v] & \subseteq & [\![k]\!]^\sharp\,(\mathcal{A}[u]) \qquad\qquad k = (u, \_, v) \quad \text{edge}
\end{array}
$$

Wanted:

- a maximally large solution (??)
- an algorithm which computes this :-)

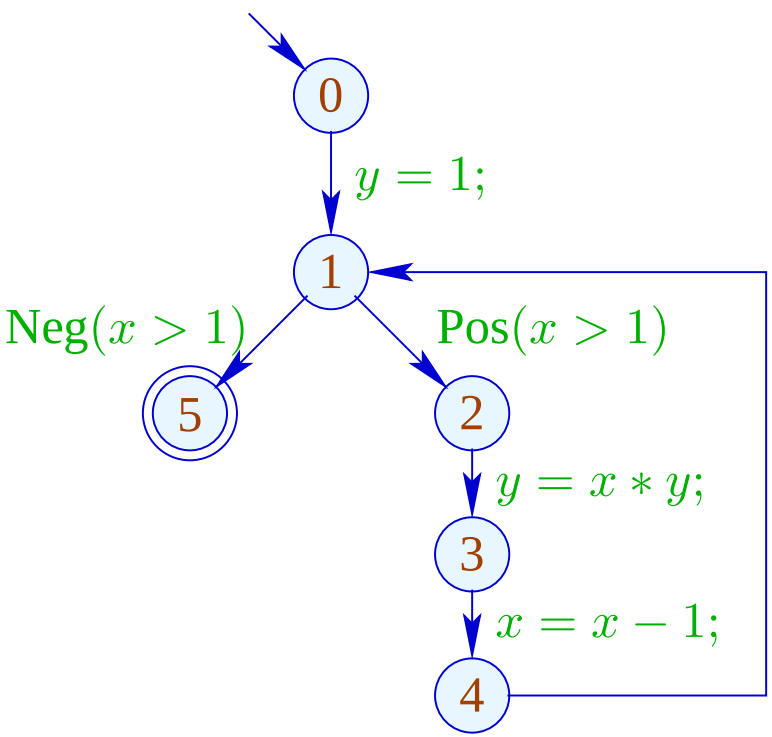
Example:



Wanted:

- a maximally large solution (??)
- an algorithm which computes this :-)

Example:

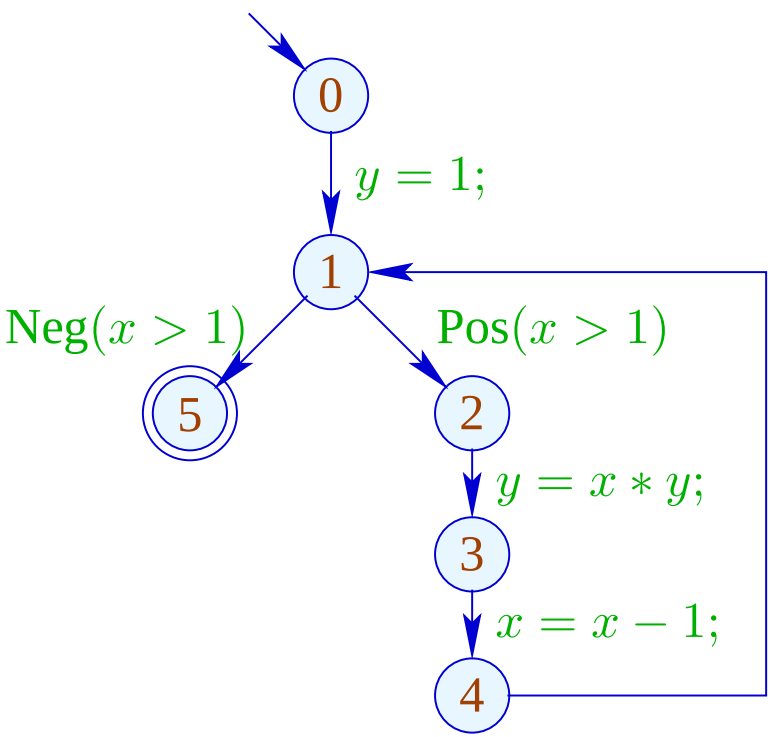


$$\mathcal{A}[0] \quad \subseteq \quad \emptyset$$

Wanted:

- a maximally large solution (??)
- an algorithm which computes this :-)

Example:

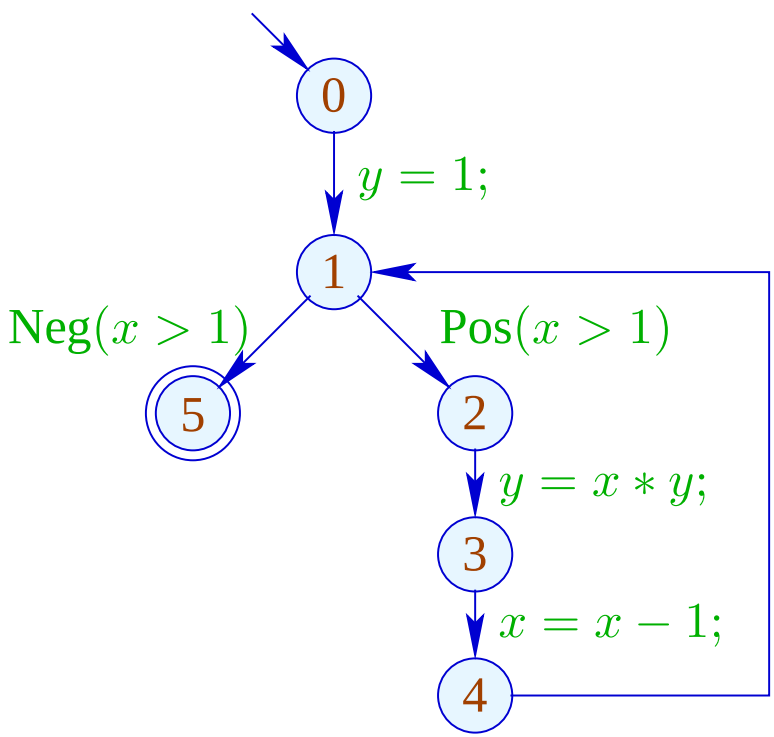


$$\begin{aligned}
\mathcal{A}[0] &\subseteq \emptyset \\
\mathcal{A}[1] &\subseteq (\mathcal{A}[0] \cup \{1\}) \backslash \mathit{Expr}_y \\
\mathcal{A}[1] &\subseteq \mathcal{A}[4]
\end{aligned}$$

Wanted:

- a maximally large solution (??)
- an algorithm which computes this :-)

Example:

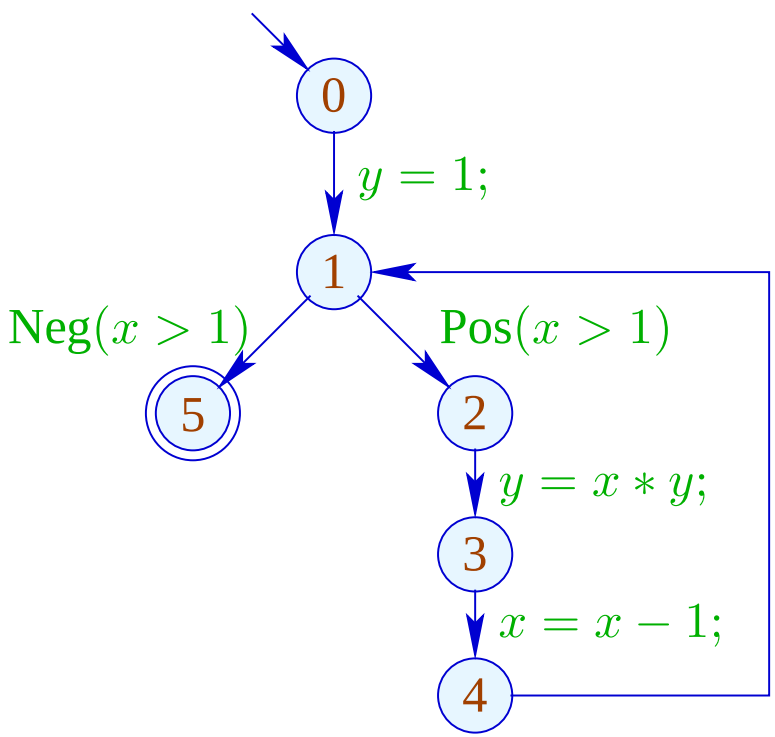


$$
\begin{aligned}
\mathcal{A}[0] &\subseteq \emptyset \\
\mathcal{A}[1] &\subseteq (\mathcal{A}[0] \cup \{1\}) \backslash \mathit{Expr}_y \\
\mathcal{A}[1] &\subseteq \mathcal{A}[4] \\
\mathcal{A}[2] &\subseteq \mathcal{A}[1] \cup \{x > 1\}
\end{aligned}
$$

Wanted:

- a maximally large solution (??)
- an algorithm which computes this :-)

Example:

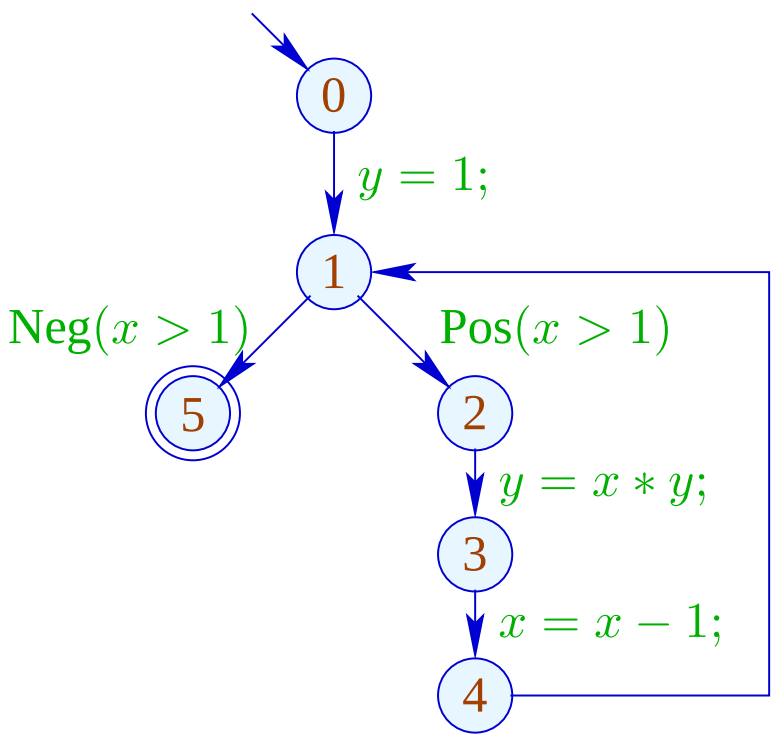


$$
\begin{array}{lcl}
\mathcal{A}[0] & \subseteq & \emptyset \\
\mathcal{A}[1] & \subseteq & (\mathcal{A}[0] \cup \{1\}) \backslash Expr_y \\
\mathcal{A}[1] & \subseteq & \mathcal{A}[4] \\
\mathcal{A}[2] & \subseteq & \mathcal{A}[1] \cup \{x > 1\} \\
\mathcal{A}[3] & \subseteq & (\mathcal{A}[2] \cup \{x * y\}) \backslash Expr_y
\end{array}
$$

Wanted:

- a maximally large solution (??)
- an algorithm which computes this :-)

Example:

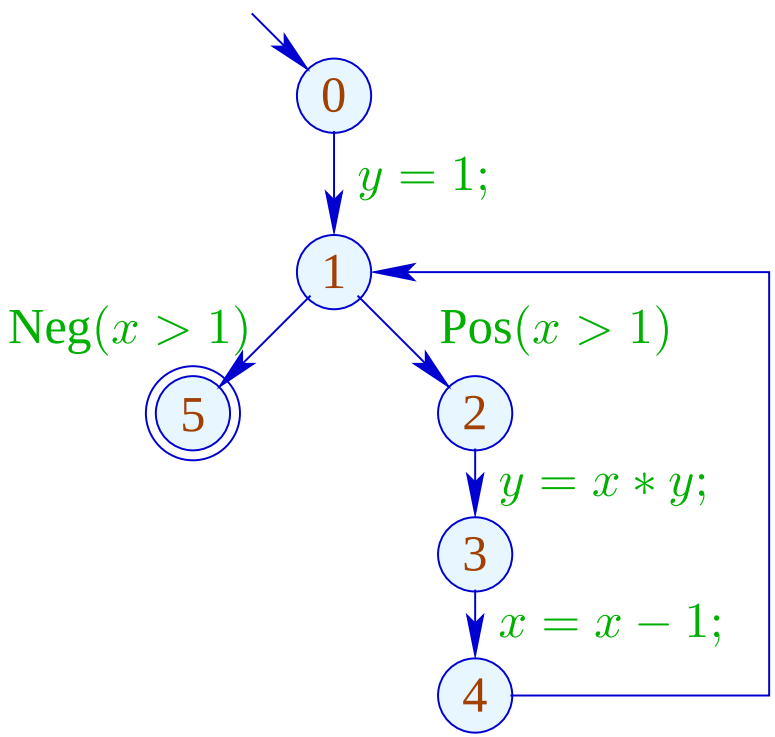


$$
\begin{array}{lcl}
\mathcal{A}[0] & \subseteq & \emptyset \\
\mathcal{A}[1] & \subseteq & (\mathcal{A}[0] \cup \{1\}) \backslash Expr_y \\
\mathcal{A}[1] & \subseteq & \mathcal{A}[4] \\
\mathcal{A}[2] & \subseteq & \mathcal{A}[1] \cup \{x > 1\} \\
\mathcal{A}[3] & \subseteq & (\mathcal{A}[2] \cup \{x * y\}) \backslash Expr_y \\
\mathcal{A}[4] & \subseteq & (\mathcal{A}[3] \cup \{x - 1\}) \backslash Expr_x
\end{array}
$$

Wanted:

- a maximally large solution (??)
- an algorithm which computes this :-)

Example:

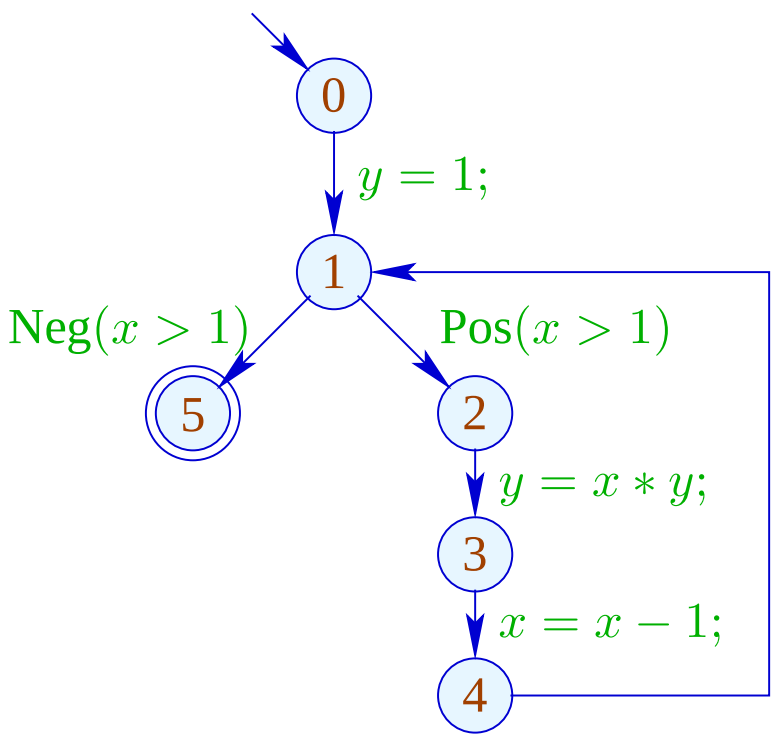


$$
\begin{array}{lcl}
\mathcal{A}[0] & \subseteq & \emptyset \\
\mathcal{A}[1] & \subseteq & (\mathcal{A}[0] \cup \{1\}) \backslash Expr_y \\
\mathcal{A}[1] & \subseteq & \mathcal{A}[4] \\
\mathcal{A}[2] & \subseteq & \mathcal{A}[1] \cup \{x > 1\} \\
\mathcal{A}[3] & \subseteq & (\mathcal{A}[2] \cup \{x * y\}) \backslash Expr_y \\
\mathcal{A}[4] & \subseteq & (\mathcal{A}[3] \cup \{x - 1\}) \backslash Expr_x \\
\mathcal{A}[5] & \subseteq & \mathcal{A}[1] \cup \{x > 1\}
\end{array}
$$

Wanted:

- a maximally large solution (??)
- an algorithm which computes this :-)

Example:



Solution:

$$
\begin{aligned}
\mathcal{A}[0] &= \emptyset \\
\mathcal{A}[1] &= \{1\} \\
\mathcal{A}[2] &= \{1, x > 1\} \\
\mathcal{A}[3] &= \{1, x > 1\} \\
\mathcal{A}[4] &= \{1\} \\
\mathcal{A}[5] &= \{1, x > 1\}
\end{aligned}
$$

Observation:

- The possible values for $\mathcal{A}[u]$ form a complete lattice:

$$\mathbb{D} = 2^{Expr} \quad \text{with} \quad B_1 \sqsubseteq B_2 \quad \text{iff} \quad B_1 \supseteq B_2$$

Observation:

- The possible values for $\mathcal{A}[u]$ form a complete lattice:

$$\mathbb{D} = 2^{Expr} \quad \text{with} \quad B_1 \sqsubseteq B_2 \quad \text{iff} \quad B_1 \supseteq B_2$$

- The functions $[\![k]\!]^\sharp : \mathbb{D} \to \mathbb{D}$ are monotonic, i.e.,

$$[\![k]\!]^\sharp(B_1) \sqsubseteq [\![k]\!]^\sharp(B_2) \quad \text{iff} \quad B_1 \sqsubseteq B_2$$

Background 2: Complete Lattices

A set $\mathbb{D}$ together with a relation $\sqsubseteq \subseteq \mathbb{D} \times \mathbb{D}$ is a partial order if for all $a, b, c \in \mathbb{D}$,

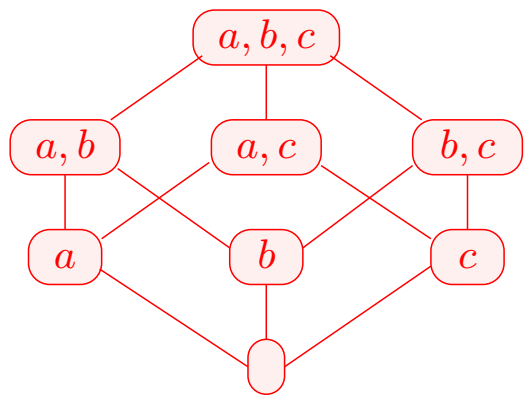
$$a \sqsubseteq a \qquad \textit{reflexivity}$$

$$a \sqsubseteq b \wedge b \sqsubseteq a \implies a = b \qquad \textit{anti-symmetry}$$

$$a \sqsubseteq b \wedge b \sqsubseteq c \implies a \sqsubseteq c \qquad \textit{transitivity}$$

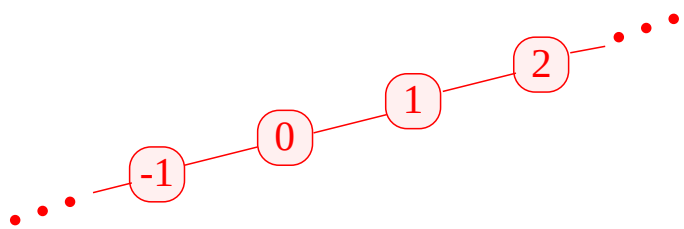
Examples:

1. $\mathbb{D} = 2^{\{a,b,c\}}$ with the relation "$\subseteq$" :

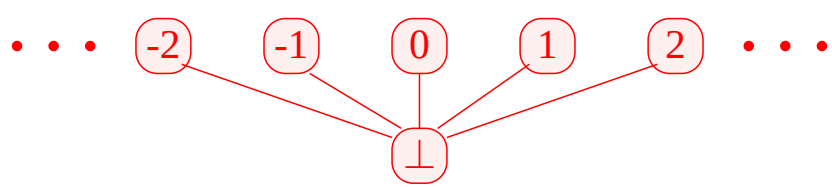


2. $\mathbb{Z}$ with the relation "$=$" :

3. $\mathbb{Z}$ with the relation "$\leq$" :



4. $\mathbb{Z}_\perp = \mathbb{Z} \cup \{\perp\}$ with the ordering:



$d \in \mathbb{D}$ is called upper bound for $X \subseteq \mathbb{D}$ if

$$x \sqsubseteq d \qquad \text{for all } x \in X$$

$d \in \mathbb{D}$ is called upper bound for $X \subseteq \mathbb{D}$ if

$$x \sqsubseteq d \qquad \text{for all } x \in X$$

d is called least upper bound (lub) if

1. d is an upper bound and
2. $d \sqsubseteq y$ for every upper bound y of X.

$d \in \mathbb{D}$ is called upper bound for $X \subseteq \mathbb{D}$ if

$$x \sqsubseteq d \qquad \text{for all } x \in X$$

d is called least upper bound (lub) if

1. d is an upper bound and
2. $d \sqsubseteq y$ for every upper bound y of X.

Caveat:

- $\{0, 2, 4, \ldots\} \subseteq \mathbb{Z}$ has no upper bound!
- $\{0, 2, 4\} \subseteq \mathbb{Z}$ has the upper bounds $4, 5, 6, \ldots$

A complete lattice (cl) $\mathbb{D}$ is a partial ordering where every subset $X \subseteq \mathbb{D}$ has a least upper bound $\bigsqcup X \in \mathbb{D}$.

Note:

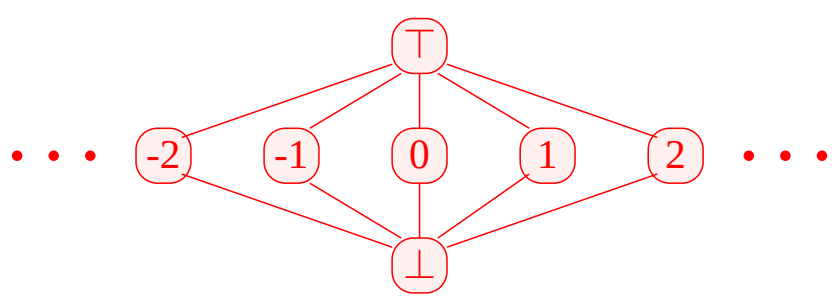
Every complete lattice has

$\rightarrow$ a least element $\bot = \bigsqcup \emptyset \in \mathbb{D}$;

$\rightarrow$ a greatest element $\top = \bigsqcup \mathbb{D} \in \mathbb{D}$.

Examples:

1. $\mathbb{D} = 2^{\{a,b,c\}}$ is a cl :-)
2. $\mathbb{D} = \mathbb{Z}$ with "$=$" is not.
3. $\mathbb{D} = \mathbb{Z}$ with "$\leq$" is neither.
4. $\mathbb{D} = \mathbb{Z}_\perp$ is also not :-(
5. With an extra element $\top$, we obtain the flat lattice $\mathbb{Z}_\perp^\top = \mathbb{Z} \cup \{\perp, \top\}$:



We have:

Theorem:

If $\mathbb{D}$ is a complete lattice, then every subset $X \subseteq \mathbb{D}$ has a greatest lower bound $\bigsqcap X$.

We have:

Theorem:

If $\mathbb{D}$ is a complete lattice, then every subset $X \subseteq \mathbb{D}$ has a greatest lower bound $\bigsqcap X$.

Proof:

Construct $U = \{u \in \mathbb{D} \mid \forall\, x \in X : \; u \sqsubseteq x\}$.

// the set of all lower bounds of X :-)

We have:

Theorem:

If $\mathbb{D}$ is a complete lattice, then every subset $X \subseteq \mathbb{D}$ has a greatest lower bound $\bigsqcap X$.

Proof:

Construct $U = \{u \in \mathbb{D} \mid \forall\, x \in X : \; u \sqsubseteq x\}$.

// the set of all lower bounds of X :-)

Set: $g := \bigsqcup U$

Claim: $g = \bigsqcap X$

(1) g is a lower bound of X :

Assume $x \in X$. Then:

$u \sqsubseteq x$ for all $u \in U$

$\Longrightarrow$ x is an upper bound of U

$\Longrightarrow$ $g \sqsubseteq x$:-)

(1) g is a lower bound of X :

Assume $x \in X$. Then:

$u \sqsubseteq x$ for all $u \in U$

$\Longrightarrow$ x is an upper bound of U

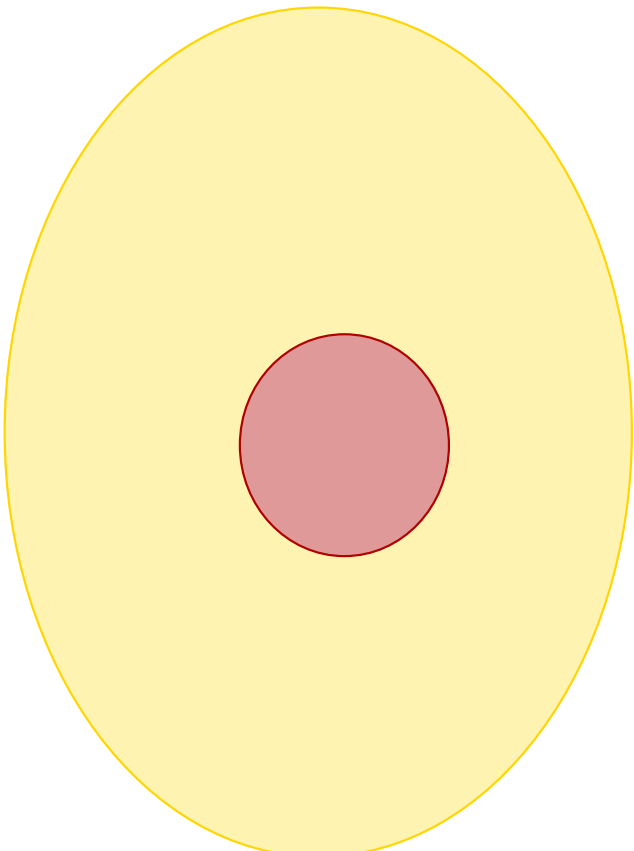
$\Longrightarrow$ $g \sqsubseteq x$:-)

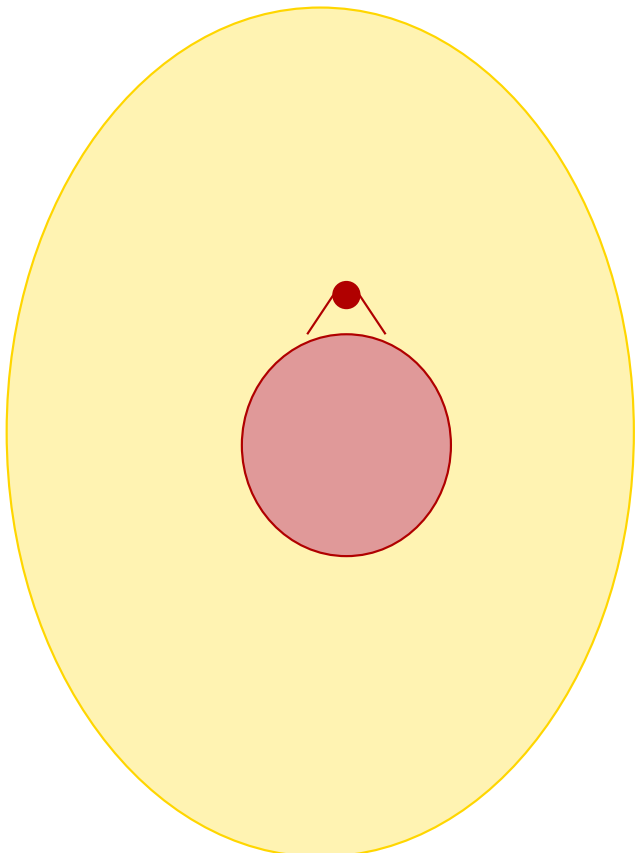
(2) g is the greatest lower bound of X :

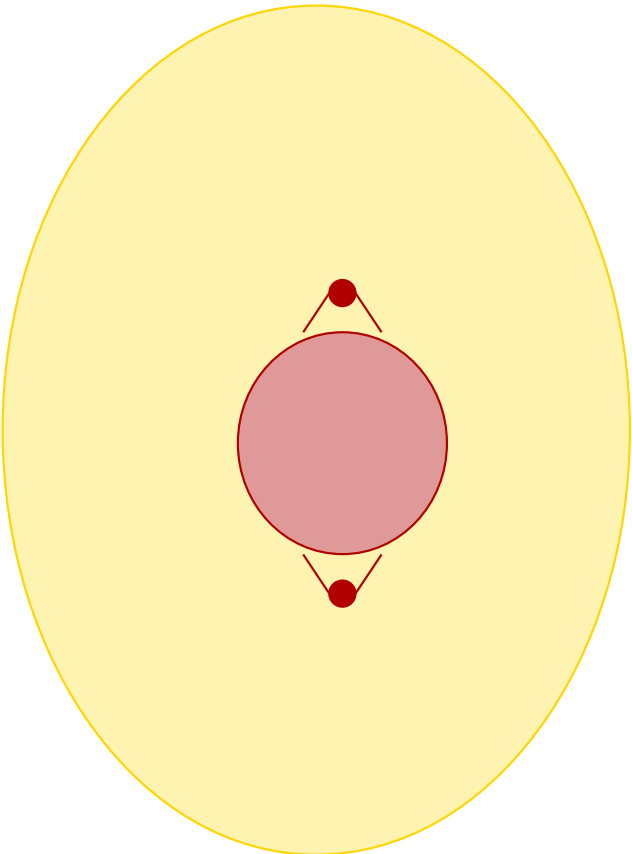
Assume u is a lower bound of X. Then:

$u \in U$

$\Longrightarrow$ $u \sqsubseteq g$:-))







We are looking for solutions for systems of constraints of the form:

$$x_i \quad \sqsupseteq \quad f_i(x_1, \ldots, x_n) \qquad\qquad (*)$$

We are looking for solutions for systems of constraints of the form:

$$x_i \quad \sqsupseteq \quad f_i(x_1, \ldots, x_n) \qquad\qquad (*)$$

where:

x_i	unknown	here:	$\mathcal{A}[u]$
$\mathbb{D}$	values	here:	2^{Expr}
$\sqsubseteq \;\subseteq\; \mathbb{D} \times \mathbb{D}$	ordering relation	here:	$\supseteq$
$f_i\colon \mathbb{D}^n \to \mathbb{D}$	constraint	here:	...

We are looking for solutions for systems of constraints of the form:

$$x_i \quad \sqsupseteq \quad f_i(x_1, \ldots, x_n) \qquad\qquad (*)$$

where:

x_i	unknown	here:	$\mathcal{A}[u]$
$\mathbb{D}$	values	here:	2^{Expr}
$\sqsubseteq \;\subseteq\; \mathbb{D} \times \mathbb{D}$	ordering relation	here:	$\supseteq$
$f_i\colon \mathbb{D}^n \to \mathbb{D}$	constraint	here:	...

Constraint for $\mathcal{A}[v] \quad (v \neq start)$:

$$\mathcal{A}[v] \quad \subseteq \quad \bigcap \{ [\![k]\!]^\sharp (\mathcal{A}[u]) \mid k = (u, \_, v) \;\; \text{edge} \}$$

We are looking for solutions for systems of constraints of the form:

$$x_i \quad \sqsupseteq \quad f_i(x_1, \ldots, x_n) \qquad\qquad (*)$$

where:

x_i	unknown	here:	$\mathcal{A}[u]$
$\mathbb{D}$	values	here:	2^{Expr}
$\sqsubseteq \;\subseteq\; \mathbb{D} \times \mathbb{D}$	ordering relation	here:	$\supseteq$
$f_i\colon \mathbb{D}^n \to \mathbb{D}$	constraint	here:	...

Constraint for $\mathcal{A}[v] \quad (v \neq start)$:

$$\mathcal{A}[v] \quad \subseteq \quad \bigcap \{ [\![k]\!]^\sharp (\mathcal{A}[u]) \mid k = (u, \_, v) \;\; \text{edge} \}$$

Because:

$$x \sqsupseteq d_1 \wedge \ldots \wedge x \sqsupseteq d_k \quad \text{iff} \quad x \sqsupseteq \bigsqcup \{d_1, \ldots, d_k\} \qquad \text{:-)}$$

A mapping $f : \mathbb{D}_1 \to \mathbb{D}_2$ is called monotonic, if $f(a) \sqsubseteq f(b)$ for all $a \sqsubseteq b$.

A mapping $f : \mathbb{D}_1 \to \mathbb{D}_2$ is called monotonic, if $f(a) \sqsubseteq f(b)$ for all $a \sqsubseteq b$.

Examples:

(1) $\mathbb{D}_1 = \mathbb{D}_2 = 2^U$ for a set U and $f\,x = (x \cap a) \cup b$.

Obviously, every such f is monotonic :-)

A mapping $f : \mathbb{D}_1 \to \mathbb{D}_2$ is called monotonic, is $f(a) \sqsubseteq f(b)$ for all $a \sqsubseteq b$.

Examples:

(1) $\mathbb{D}_1 = \mathbb{D}_2 = 2^U$ for a set U and $f\,x = (x \cap a) \cup b$.

Obviously, every such f is monotonic :-)

(2) $\mathbb{D}_1 = \mathbb{D}_2 = \mathbb{Z}$ (with the ordering "$\leq$"). Then:

- $\mathsf{inc}\,x = x + 1$ is monotonic.
- $\mathsf{dec}\,x = x - 1$ is monotonic.

A mapping $f : \mathbb{D}_1 \to \mathbb{D}_2$ is called monotonic, is $f(a) \sqsubseteq f(b)$ for all $a \sqsubseteq b$.

Examples:

(1) $\mathbb{D}_1 = \mathbb{D}_2 = 2^U$ for a set U and $f\,x = (x \cap a) \cup b$.

Obviously, every such f is monotonic :-)

(2) $\mathbb{D}_1 = \mathbb{D}_2 = \mathbb{Z}$ (with the ordering "$\leq$"). Then:

- $\mathsf{inc}\,x = x + 1$ is monotonic.
- $\mathsf{dec}\,x = x - 1$ is monotonic.
- $\mathsf{inv}\,x = -x$ is not monotonic :-)

Theorem:

If $f_1 : \mathbb{D}_1 \to \mathbb{D}_2$ and $f_2 : \mathbb{D}_2 \to \mathbb{D}_3$ are monotonic, then also $f_2 \circ f_1 : \mathbb{D}_1 \to \mathbb{D}_3$:-)