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If $f_1 : \mathbb{D}_1 \to \mathbb{D}_2$ and $f_2 : \mathbb{D}_2 \to \mathbb{D}_3$ are monotonic, then also $f_2 \circ f_1 : \mathbb{D}_1 \to \mathbb{D}_3$:-)

Theorem:

If $\mathbb{D}_2$ is a complete lattice, then the set $[\mathbb{D}_1 \to \mathbb{D}_2]$ of monotonic functions $f : \mathbb{D}_1 \to \mathbb{D}_2$ is also a complete lattice where

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In particular for $F \subseteq [\mathbb{D}_1 \to \mathbb{D}_2]$,

$$\bigsqcup F = f \quad \text{mit} \quad f\,x = \bigsqcup \{g\,x \mid g \in F\}$$

For functions $f_i\,x = a_i \cap x \cup b_i$, the operations "$\circ$", "$\sqcup$" and "$\sqcap$" can be explicitly defined by:

$$
\begin{aligned}
(f_2 \circ f_1)\,x &= \boxed{a_1 \cap a_2} \cap x \cup \boxed{a_2 \cap b_1 \cup b_2} \\
(f_1 \sqcup f_2)\,x &= \boxed{(a_1 \cup a_2)} \cap x \cup \boxed{b_1 \cup b_2} \\
(f_1 \sqcap f_2)\,x &= \boxed{(a_1 \cup b_1) \cap (a_2 \cup b_2)} \cap x \cup \boxed{b_1 \cap b_2}
\end{aligned}
$$

Wanted: minimally small solution for:

$$x_i \sqsupseteq f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n \qquad (*)$$

where all $f_i : \mathbb{D}^n \to \mathbb{D}$ are monotonic.

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Idea:

- Consider $F : \mathbb{D}^n \to \mathbb{D}^n$ where

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- If all f_i are monotonic, then also F :-)
- We successively approximate a solution. We construct:

  $$\underline{\perp}, \quad F\,\underline{\perp}, \quad F^2\,\underline{\perp}, \quad F^3\,\underline{\perp}, \quad \ldots$$

Hope: We eventually reach a solution ... ???

Example: $\mathbb{D} = 2^{\{a,b,c\}}, \quad \sqsubseteq \, = \, \subseteq$

$$\begin{array}{lll} x_1 & \supseteq & \{a\} \cup x_3 \\ x_2 & \supseteq & x_3 \cap \{a, b\} \\ x_3 & \supseteq & x_1 \cup \{c\} \end{array}$$

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The Iteration:

	0	1	2	3	4
x_1	$\emptyset$				
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Theorem

- $\underline{\perp}, F\,\underline{\perp}, F^2\,\underline{\perp}, \ldots$ form an ascending chain :

$$\underline{\perp} \quad \sqsubseteq \quad F\,\underline{\perp} \quad \sqsubseteq \quad F^2\,\underline{\perp} \quad \sqsubseteq \quad \ldots$$

- If $F^k\,\underline{\perp} = F^{k+1}\,\underline{\perp}$, a solution is obtained which is the least one :-)
- If all ascending chains are finite, such a k always exists.

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Proof

The first claim follows by complete induction:

Foundation: $F^0\,\underline{\perp} = \underline{\perp} \sqsubseteq F^1\,\underline{\perp}$:-)

Step: Assume $F^{i-1}\,\underline{\perp} \;\sqsubseteq\; F^{i}\,\underline{\perp}$. Then

$$F^{i}\,\underline{\perp} = F\,(F^{i-1}\,\underline{\perp}) \sqsubseteq F\,(F^{i}\,\underline{\perp}) = F^{i+1}\,\underline{\perp}$$

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Conclusion:

If $\mathbb{D}$ is finite, a solution can be found which is definitely the least :-)

Question:

What, if $\mathbb{D}$ is not finite ???

Theorem Knaster – Tarski

Assume $\mathbb{D}$ is a complete lattice. Then every monotonic function $f : \mathbb{D} \to \mathbb{D}$ has a least fixpoint $d_0 \in \mathbb{D}$.

Let $P = \{d \in \mathbb{D} \mid f\, d \sqsubseteq d\}$.

Then $d_0 = \bigsqcap P$.

Bronisław Knaster (1893-1980), topology

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Then $d_0 = \bigsqcap P$.

Proof:

(1) $d_0 \in P$:

$$
\begin{array}{ll}
 & f\,d_0 \sqsubseteq f\,d \sqsubseteq d \quad \text{for all } d \in P \\
\Longrightarrow & f\,d_0 \quad \text{is a lower bound of } P \\
\Longrightarrow & f\,d_0 \sqsubseteq d_0 \quad \text{since } d_0 = \bigsqcap P \\
\Longrightarrow & d_0 \in P \qquad\qquad \text{:-)}
\end{array}
$$

(2) $\qquad f\, d_0 = d_0 :$

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(3) $\quad d_0$ is least fixpoint:

$$
\begin{array}{ll}
 & f\, d_1 = d_1 \sqsubseteq d_1 \quad \text{an other fixpoint} \\
\Longrightarrow & d_1 \in P \\
\Longrightarrow & d_0 \sqsubseteq d_1 \qquad \text{:-))}
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Remark:

The least fixpoint d_0 is in P and a lower bound :-)

$\Longrightarrow$ d_0 is the least value x with $x \sqsupseteq f\,x$

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Assume $x_i \sqsupseteq f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n$ $(*)$

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is a system of constraints where all $f_i : \mathbb{D}^n \to \mathbb{D}$ are monotonic.

$\Longrightarrow$ least solution of $(*)$ $==$ least fixpoint of F :-)

Example 1: $\mathbb{D} = 2^U, \quad f\,x = x \cap a \cup b$

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Example 2: $\mathbb{D} = \mathbb{N} \cup \{\infty\}$

Assume $f\,x = x + 1$. Then

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Assume $f\,x = x + 1$. Then

$$f^i\,\bot = f^i\,0 = i \quad \sqsubset \quad i + 1 = f^{i+1}\,\bot$$

$\Longrightarrow$ Ordinary iteration will never reach a fixpoint :-(

$\Longrightarrow$ Sometimes, transfinite iteration is needed :-)

Conclusion:

Systems of inequations can be solved through fixpoint iteration, i.e., by repeated evaluation of right-hand sides   :-)

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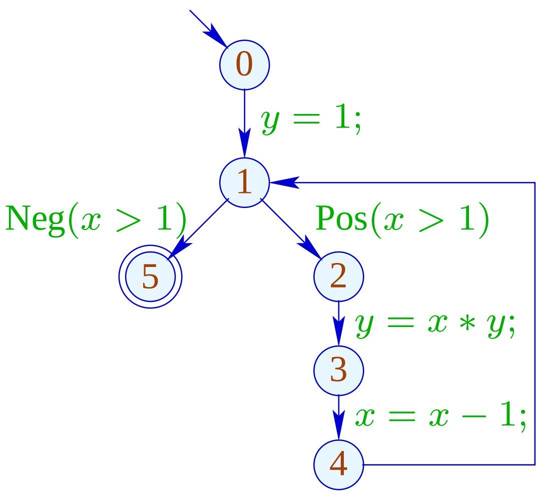

0
1
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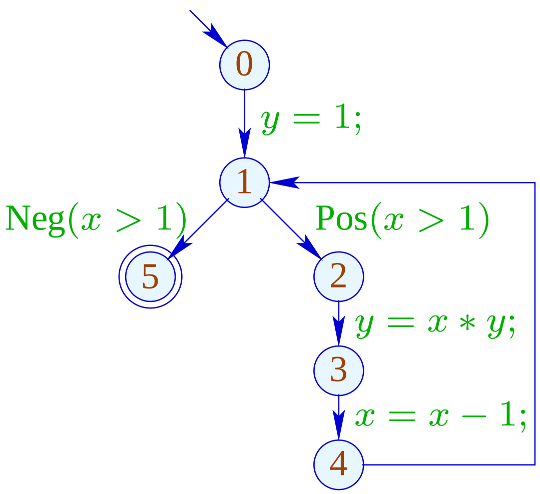

	1
0	$\emptyset$
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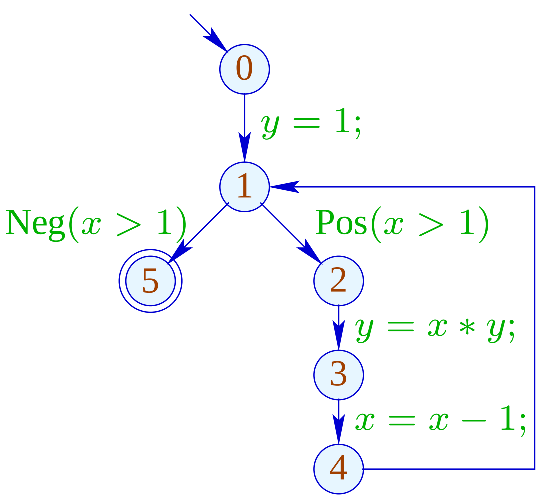

	1	2
0	$\emptyset$	$\emptyset$
1	$\{1, x > 1, x - 1\}$	$\{1\}$
2	$Expr$	$\{1, x > 1, x - 1\}$
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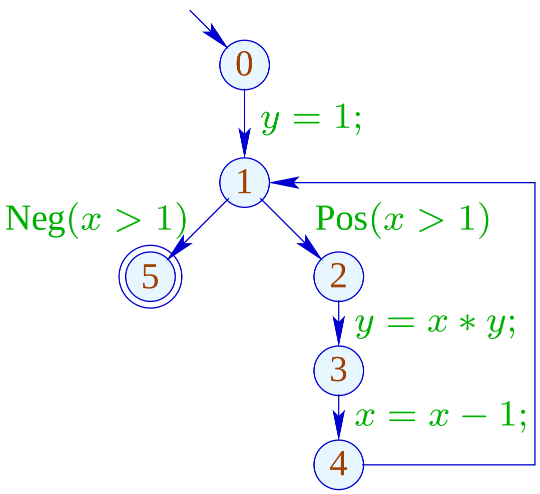

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2	$Expr$	$\{1, x > 1, x - 1\}$	$\{1, x > 1\}$
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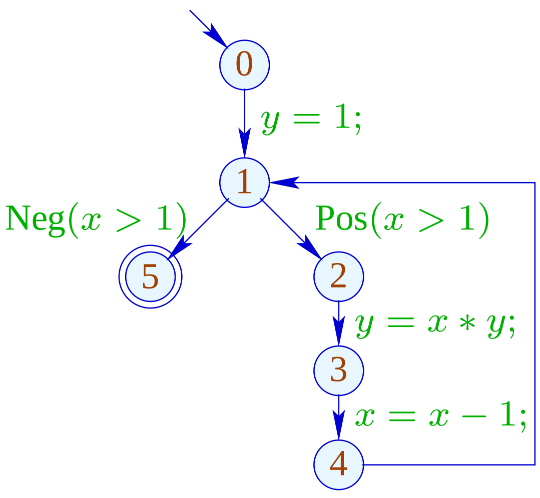

	1	2	3	4
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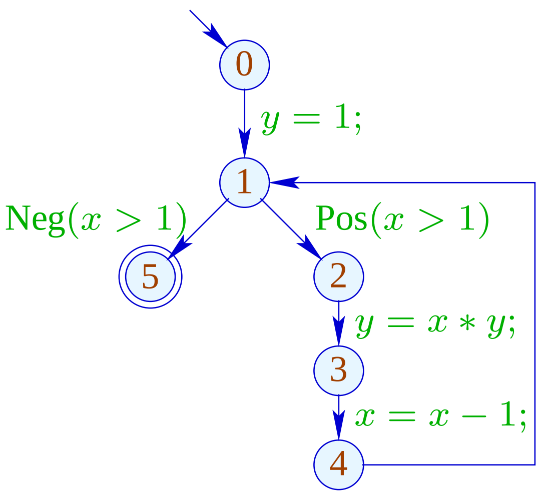

	1	2	3	4	5
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	
1	$\{1, x > 1, x - 1\}$	$\{1\}$	$\{1\}$	$\{1\}$	
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3	$\{1, x > 1, x - 1\}$	$\{1, x > 1, x - 1\}$	$\{1, x > 1, x - 1\}$	$\{1, x > 1\}$	dito
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Instead of accessing the values of the last iteration, always use the current values of unknowns :-)

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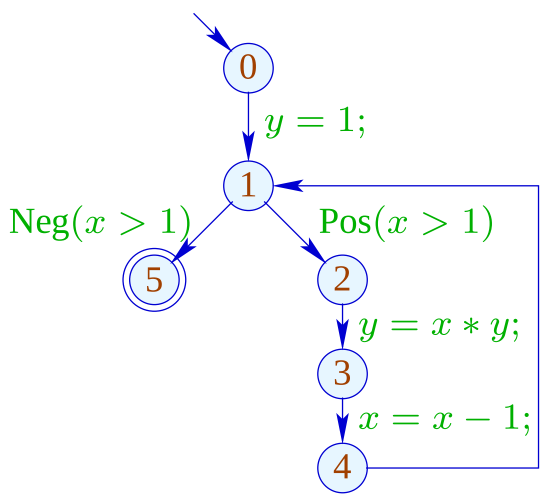

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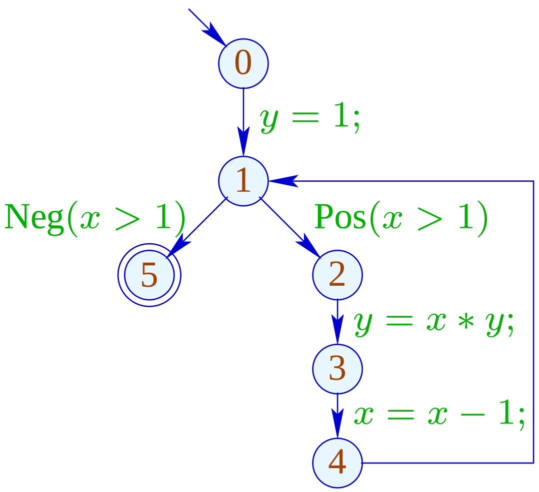

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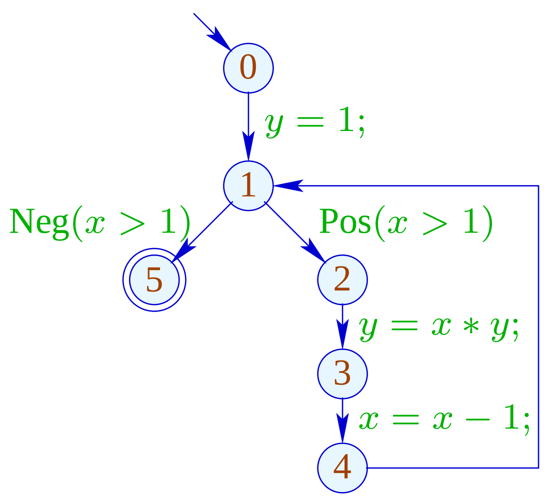

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The code for Round Robin Iteration in Java looks as follows:

$$
\begin{array}{l}
\textsf{for}\ (i = 1; i \leq n; i{+}{+})\ x_i = \bot; \\
\textsf{do}\ \{ \\
\qquad \mathit{finished} = \textsf{true}; \\
\qquad \textsf{for}\ (i = 1; i \leq n; i{+}{+})\ \{ \\
\qquad\qquad \mathit{new} = f_i(x_1, \ldots, x_n); \\
\qquad\qquad \textsf{if}\ (!(x_i \sqsupseteq \mathit{new}))\ \{ \\
\qquad\qquad\qquad \mathit{finished} = \textsf{false}; \\
\qquad\qquad\qquad x_i = x_i \sqcup \mathit{new}; \\
\qquad\qquad \} \\
\qquad \} \\
\}\ \textsf{while}\ (!\mathit{finished});
\end{array}
$$