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Caveat:

The efficiency of RR-iteration depends on the ordering of the unknowns !!!

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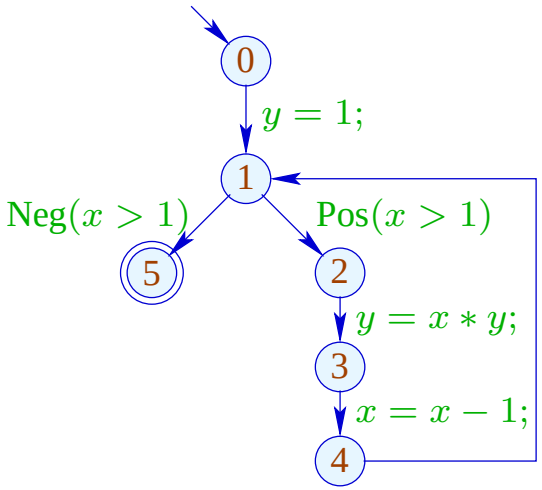
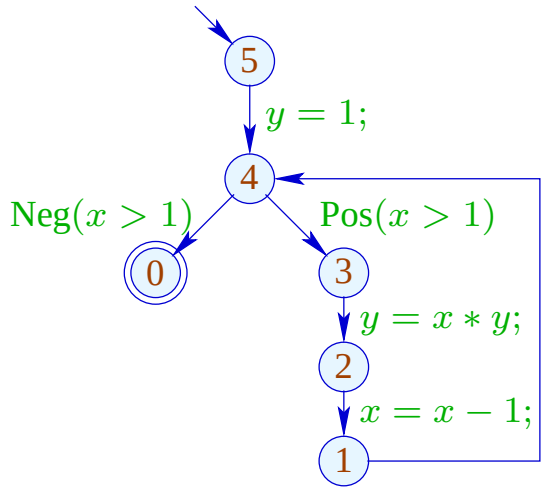
Bad:

e.g., post-order DFS of the CFG, starting at start :-)

Good:


0
$y = 1;$
1
$\text{Neg}(x > 1)$
$\text{Pos}(x > 1)$
5
2
$y = x * y;$
3
$x = x - 1;$
4

Bad: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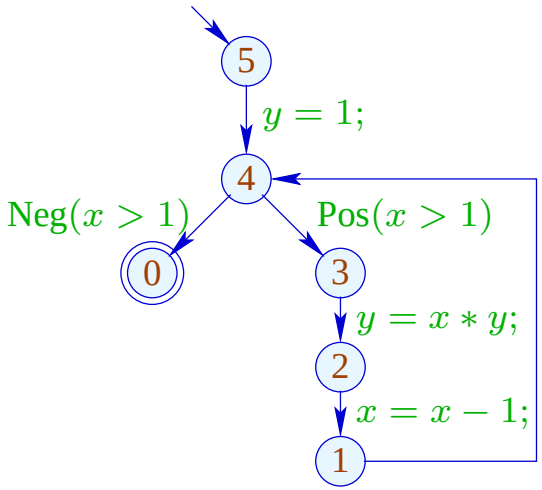
Inefficient Round Robin Iteration:



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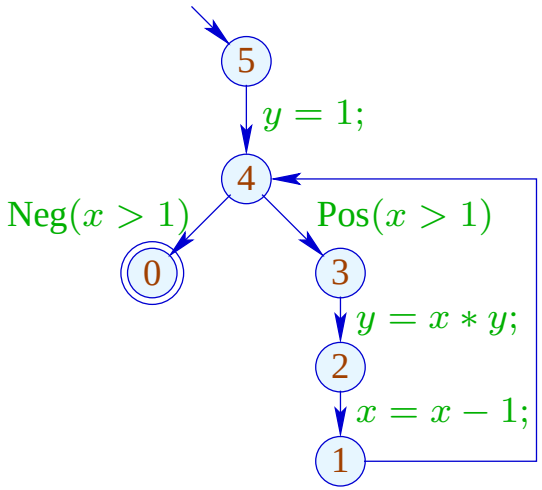
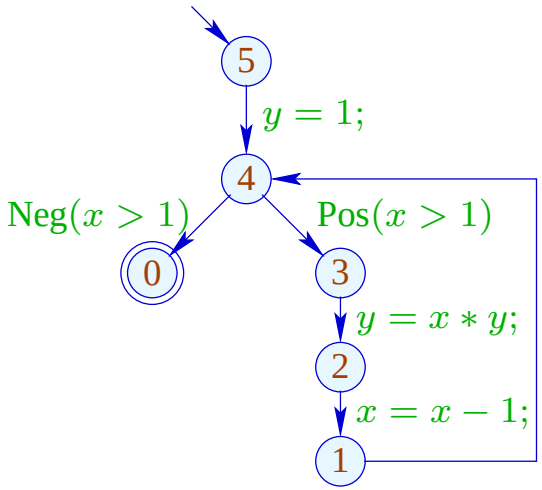
Inefficient Round Robin Iteration:



	1
0	*Expr*
1	$\{1\}$
2	$\{1, x-1, x>1\}$
3	*Expr*
4	$\{1\}$
5	$\emptyset$

Inefficient Round Robin Iteration:



	1	2
0	*Expr*	$\{1, x > 1\}$
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2	$\{1, x - 1, x > 1\}$	$\{1, x - 1, x > 1\}$
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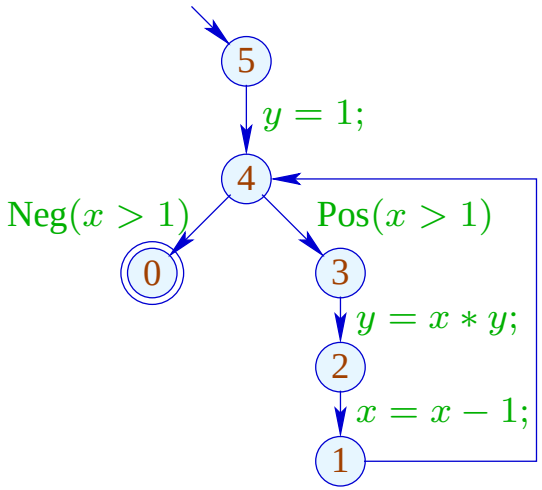

	1	2	3
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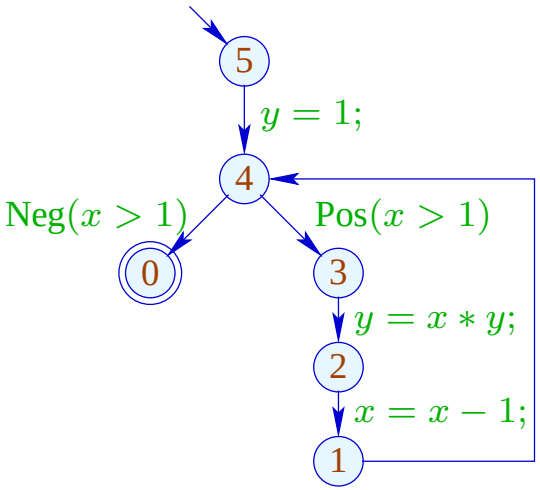

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2	$\{1, x - 1, x > 1\}$	$\{1, x - 1, x > 1\}$	$\{1, x > 1\}$	dito
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$\Longrightarrow$ significantly less efficient :-)

... end of background on: Complete Lattices

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For a complete lattice $\mathbb{D}$, consider systems:

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where $d_0 \in \mathbb{D}$ and all $[\![k]\!]^\sharp : \mathbb{D} \to \mathbb{D}$ are monotonic ...

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$\Longrightarrow$ Monotonic Analysis Framework

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Jeffrey D. Ullman, Stanford

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In particular: $\mathcal{I}[v] \sqsupseteq [\![\pi]\!]^\sharp \, d_0$ for every $\pi : start \to^* v$

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Then:

$$\begin{array}{rlll} & [\![\pi']\!]^\sharp\, d_0 & \sqsubseteq & \mathcal{I}[u] \qquad \text{by I.H. for } \pi \\ \Longrightarrow & [\![\pi]\!]^\sharp\, d_0 & = & [\![k]\!]^\sharp\, ([\![\pi']\!]^\sharp\, d_0) \\ & & \sqsubseteq & [\![k]\!]^\sharp\, (\mathcal{I}[u]) \qquad \text{since } [\![k]\!]^\sharp \text{ monotonic} \\ & & \sqsubseteq & \mathcal{I}[v] \qquad \text{since } \mathcal{I} \text{ solution :-))} \end{array}$$

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With the notable exception when all functions $[\![k]\!]^\sharp$ are distributive ... :-)

The function $f : \mathbb{D}_1 \to \mathbb{D}_2$ is called

- distributive, if $f\,(\bigsqcup X) = \bigsqcup\{f\,x \mid x \in X\}$ for all $\emptyset \neq X \subseteq \mathbb{D}$;
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  Distributivity:

$$\begin{aligned} f\,(x_1 \cup x_2) &= a \cap (x_1 \cup x_2) \cup b \\ &= a \cap x_1 \cup a \cap x_2 \cup b \\ &= f\,x_1 \cup f\,x_2 \qquad \text{:-)} \end{aligned}$$

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  Distributivity:

$$f\,((1,4) \sqcup (4,1)) = f\,(4,4) = 8$$
$$\neq 5 = f\,(1,4) \sqcup f\,(4,1) \quad :-)$$

Remark:

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From that follows:

$$\begin{aligned} f\ b &= f\ (a \sqcup b) \\ &= f\ a \sqcup f\ b \\ \Longrightarrow\ f\ a &\sqsubseteq f\ b \end{aligned}$$:-)

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Gary A. Kildall (1942-1994).

Has developed the operating system CP/M and GUIs for PCs.

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Proof:

It suffices to prove that $\mathcal{I}^*$ is a solution :-)

For this, we show that $\mathcal{I}^*$ satisfies all constraints :-))

(1) We prove for *start* :

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\mathcal{I}^*[start] &= \bigsqcup \{ [\![\pi]\!]^\sharp\, d_0 \mid \pi : start \to^* start \} \\
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\end{aligned}$$

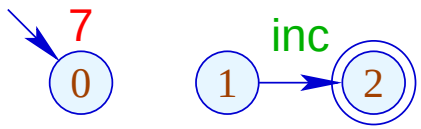
(2) For every $k = (u, \_, v)$ we prove:

$$\begin{aligned}
\mathcal{I}^*[v] &= \bigsqcup\{[\![\pi]\!]^\sharp\, d_0 \mid \pi : start \to^* v\} \\
&\sqsupseteq \bigsqcup\{[\![\pi' k]\!]^\sharp\, d_0 \mid \pi' : start \to^* u\} \\
&= \bigsqcup\{[\![k]\!]^\sharp\, ([\![\pi']\!]^\sharp\, d_0) \mid \pi' : start \to^* u\} \\
&= [\![k]\!]^\sharp\, (\bigsqcup\{[\![\pi']\!]^\sharp\, d_0 \mid \pi' : start \to^* u\}) \\
&= [\![k]\!]^\sharp\, (\mathcal{I}^*[u])
\end{aligned}$$

since $\{\pi' \mid \pi' : start \to^* u\}$ is non-empty :-)

Caveat:

- Reachability of all program points cannot be abandoned! Consider:



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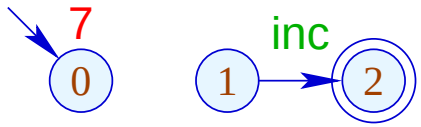

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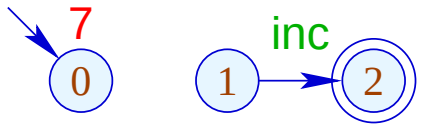

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$$\mathcal{I}[2] = \mathsf{inc}\,0 = 1$$
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- Unreachable program points can always be thrown away :-)

Summary and Application:

→ The effects of edges of the analysis of availability of expressions are distributive:

$$\begin{aligned}(a \cup (x_1 \cap x_2))\backslash b &= ((a \cup x_1) \cap (a \cup x_2))\backslash b \\ &= ((a \cup x_1)\backslash b) \cap ((a \cup x_2)\backslash b)\end{aligned}$$

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→ If all effects of edges are distributive, then the MOP can be computed by means of the constraint system and RR-iteration. :-)

→ If not all effects of edges are distributive, then RR-iteration for the constraint system at least returns a safe upper bound to the MOP :-)

1.2 Removing Assignments to Dead Variables

Example:

$$
\begin{aligned}
1: &\quad x = y + 2; \\
2: &\quad y = 5; \\
3: &\quad x = y + 3;
\end{aligned}
$$

The value of x at program points 1, 2 is over-written before it can be used.

Therefore, we call the variable x dead at these program points :-)

Note:

- → Assignments to dead variables can be removed ;-)
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Formal Definition:

The variable x is called live at u along the path π starting at u relative to a set X of variables either:

if $x \in X$ and π does not contain a definition of x; or:

if π can be decomposed into: $\pi = \pi_1 \, k \, \pi_2$ such that:

- k is a use of x ; and
- π_1 does not contain a definition of x.