Simple Case:

The two inequations have no solution over $\mathbb{Q}$.

Then they also have no solution over $\mathbb{Z}$:-)

... in Our Example:

$$\begin{array}{rclcl} x & = & i & & \\ 0 & \leq & i & = & x \\ 0 & \leq & x - 1 - i & = & -1 \end{array}$$

The second inequation has no solution :-)

Equal Signs:

If a variable x occurs in all inequations with the same sign, then there is always a solution :-(

Example:

$$\begin{array}{rcl} 0 & \leq & 13 + 7 \cdot x \\ 0 & \leq & -1 + 5 \cdot x \end{array}$$

The variable x may, e.g., be chosen as:

$$x \geq \max(-\frac{13}{7}, \frac{1}{5}) = \frac{1}{5}$$

Unequal Signs:

A variable x occurs in one inequation negative, in all others positive (if at all). Then a system can be constructed without x ...

Example:

$$\begin{array}{lll} 0 & \leq & 13 - 7 \cdot x \\ 0 & \leq & -1 + 5 \cdot x \end{array} \quad \Longleftrightarrow \quad \begin{array}{lll} x & \leq & \frac{13}{7} \\ 0 & \leq & -1 + 5 \cdot x \end{array}$$

Since $0 \leq -1 + 5 \cdot \frac{13}{7}$ the system has at least a rational solution ...

One Variable:

The inequations where x occurs positive, provide lower bounds.

The inequations where x occurs negative, provide upper bounds.

If G, L are the greatest lower and the least upper bound, respectively, then all (integer) solution are in the interval $[G, L]$:-)

Example:

$$\begin{array}{rcl} 0 & \leq & 13 - 7 \cdot x \\ 0 & \leq & -1 + 5 \cdot x \end{array} \quad \Longleftrightarrow \quad \begin{array}{rcl} x & \leq & \frac{13}{7} \\ x & \geq & \frac{1}{5} \end{array}$$

The only integer solution of the system is $x = 1$:-)

Discussion:

- Solutions only matter within the bounds to the iteration variables.
- Every integer solution there provides a conflict.
- Fusion of loops is possible if no conflicts occur :-)
- The given secial cases suffice to solve the case of two variables over $\mathbb{Q}$ and of one variable over $\mathbb{Z}$:-)
- The number of variables in the inequations corresponds to the nesting-depth of for-loops $\Longrightarrow$ in general, is quite small :-)

Discussion:

- Integer Linear Programming (ILP) can decide satisfiability of a finite set of equations/inequations over $\mathbb{Z}$ of the form:

$$\sum_{i=1}^{n} a_i \cdot x_i = b \quad \text{bzw.} \quad \sum_{i=1}^{n} a_i \cdot x_i \geq b \,, \quad a_i \in \mathbb{Z}$$

- Moreover, a (linear) cost function can be optimized :-)
- Warning: The decision problem is in general, already NP-hard !!!
- Notwithstanding that, surprisingly efficient implementations exist.
- Not just loop fusion, but also other re-organizations of loops yield ILP problems ...

Background 5: Presburger Arithmetic

Many problems in computer science can be formulated without multiplication :-)

Let us first consider two simple special cases ...

1. Linear Equations

$$\begin{array}{rcrcrcr} 2x & + & 3y & & & = & 24 \\ x & - & y & + & 5z & = & 3 \end{array}$$

Question:

- Is there a solution over $\mathbb{Q}$?
- Is there a solution over $\mathbb{Z}$?
- Is there a solution over $\mathbb{N}$?

Let us reconsider the equations:

$$\begin{array}{rcrcrcr} 2x & + & 3y & & & = & 24 \\ x & - & y & + & 5z & = & 3 \end{array}$$

Answers:

- Is there a solution over $\mathbb{Q}$? Yes
- Is there a solution over $\mathbb{Z}$? No
- Is there a solution over $\mathbb{N}$? No

Complexity:

- Is there a solution over $\mathbb{Q}$? Polynomial
- Is there a solution over $\mathbb{Z}$? Polynomial
- Is there a solution over $\mathbb{N}$? NP-hard

Solution Method for Integers:

Observation 1:

$$a_1x_1 + \ldots + a_kx_k = b \qquad (\forall\, i: \; a_i \neq 0)$$

has a solution iff

$$\text{gcd}\{a_1, \ldots, a_k\} \quad \mid \quad b$$

Example:

$$5y - 10z = 18$$

has no solution over $\mathbb{Z}$:-)

Example:

$$5y - 10z = 18$$

has no solution over $\mathbb{Z}$:-)

Observation 2:

Adding a multiple of one equation to another does not change the set of solutions :-)

Example:

$$\begin{array}{rcrcrcr} 2x & + & 3y & & & = & 24 \\ x & - & y & + & 5z & = & 3 \end{array}$$

Example:

$$\begin{array}{rcrcrcr} 2x & + & 3y & & & = & 24 \\ x & - & y & + & 5z & = & 3 \end{array}$$

$$\Longrightarrow$$

$$\begin{array}{rcrcrcr} & & 5y & - & 10z & = & 18 \\ x & - & y & + & 5z & = & 3 \end{array}$$

Observation 3:

Adding multiples of columns to another column is an invertible transformation which we keep track of in a separate matrix ...

$$
\begin{array}{ccc|ccccccc}
1 & 0 & 0 & & & 5y & - & 10z & = & 18 \\
0 & 1 & 0 & x & - & y & + & 5z & = & 3 \\
0 & 0 & 1 & & & & & & &
\end{array}
$$

$$\Longrightarrow$$

$$
\begin{array}{ccc|ccccccc}
1 & 0 & 0 & & & 5y & & & = & 18 \\
0 & 1 & 2 & x & - & y & + & 3z & = & 3 \\
0 & 0 & 1 & & & & & & &
\end{array}
$$

Observation 3:

Adding multiples of columns to another column is an invertible transformation which we keep track of in a separate matrix ...

$$\begin{array}{ccc|ccccccc} 1 & 0 & 0 & & & 5y & & & = & 18 \\ 0 & 1 & 2 & x & - & y & + & 3z & = & 3 \\ 0 & 0 & 1 & & & & & & & \end{array}$$

$$\Longrightarrow$$

$$\begin{array}{ccc|ccccccc} 1 & 0 & -3 & & & 5y & & & = & 18 \\ 0 & 1 & 2 & x & - & y & & & = & 3 \\ 0 & 0 & 1 & & & & & & & \end{array}$$

$\Longrightarrow$ triangular form !!

Observation 4:

- A special solution of a triangular system can be directly read off :-)
- All solutions of a homogeneous triangular system can be directly read off :-)
- All solutions of the original system can be recovered from the solutions of the triangular system by means of the accumulated transformation matrix:-))

Example

$$\begin{array}{ccc|c} 1 & 0 & -3 & \\ 0 & 1 & 2 & \\ 0 & 0 & 1 & \end{array} \qquad \begin{array}{rcl} 5y & = & \color{red}{15} \\ x - y & = & 3 \end{array}$$

One special solution:

$$[6, 3, 0]^\top$$

All solutions of the homogeneous system are spanned by:

$$[0, 0, 1]^\top$$

Solving over $\mathbb{N}$

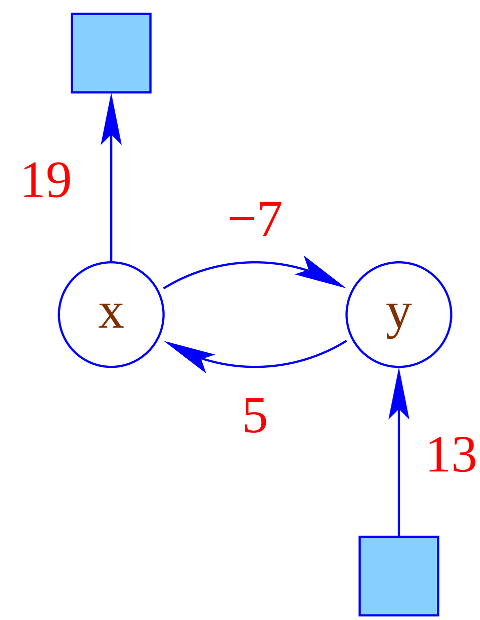
- ... is of major practical importance;
- ... has led to the development of many new techniques;
- ... easily allows to encode NP-hard problems;
- ... remains difficult if just three variables are allowed per equation.

2. One Polynomial Special Case:

$$\begin{aligned} x &\geq y + 5 \\ 19 \geq x & \\ y &\geq 13 \\ y &\geq x - 7 \end{aligned}$$

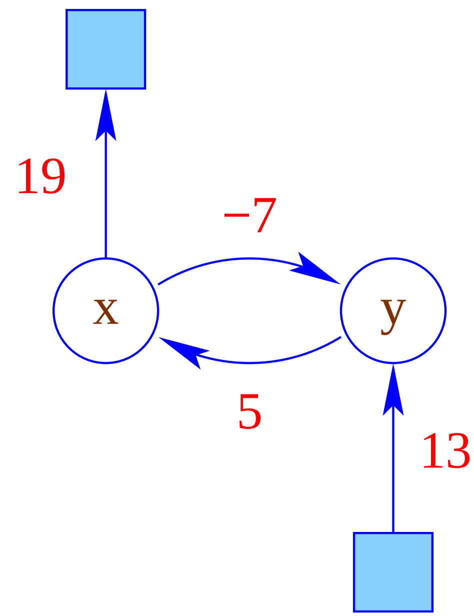
- There are at most 2 variables per in-equation;
- no scaling factors.

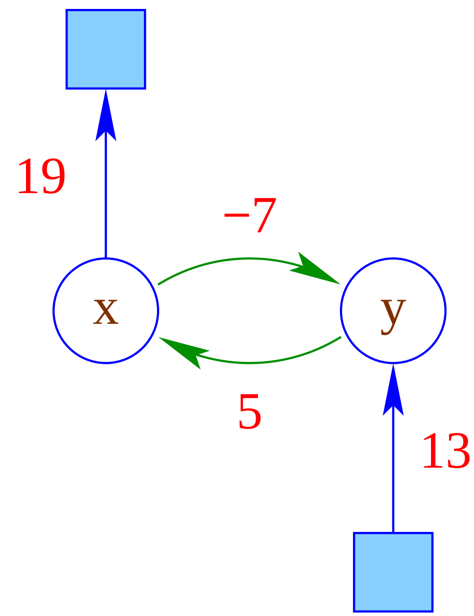
Idea: Represent the system by a graph:

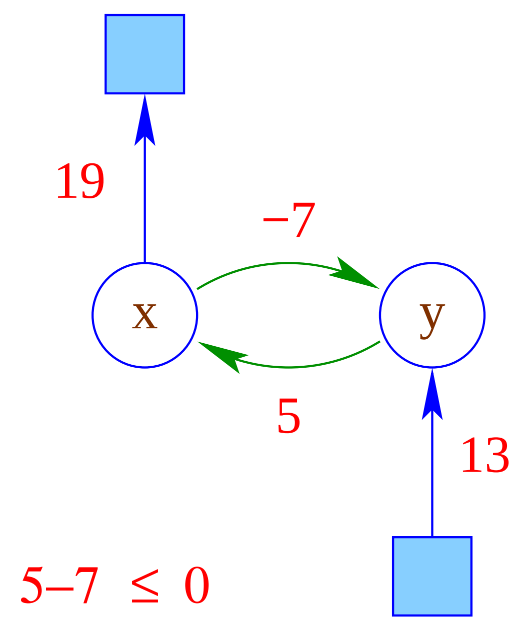


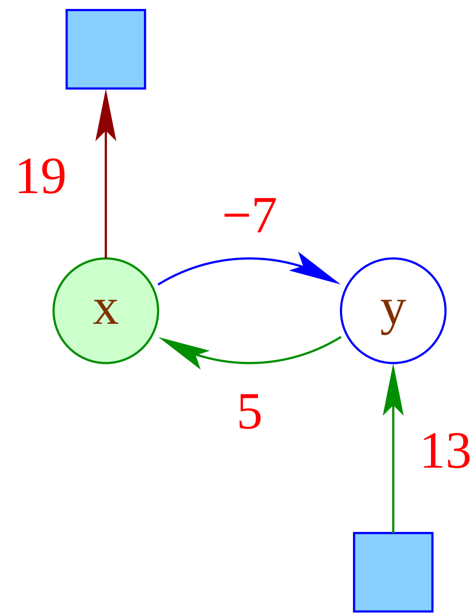
The in-equations are satisfiable iff

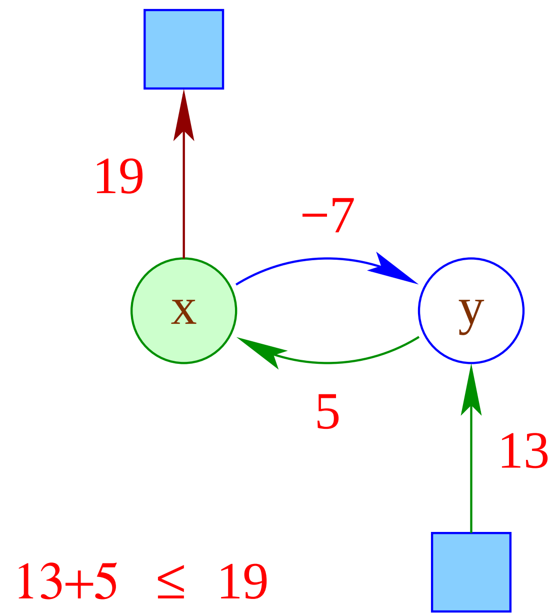
- the weight of every cycle are at most 0;
- the weights of paths reaching x are bounded by the weights leaving x.


19
−7
x
y
5
13


19
−7
x
y
5
13


19
−7
x
y
5
13
5−7 ≤ 0


19
−7
x
y
5
13


19
−7
x
y
5
13
13+5 ≤ 19

The in-equations are satisfiable iff

- the weight of every cycle are at most 0;
- the weights of paths reaching x are bounded by the weights leaving x.

$$\Longrightarrow$$

Compute the reflexive and transitive closure of the edge weights!

3. A General Solution Method:

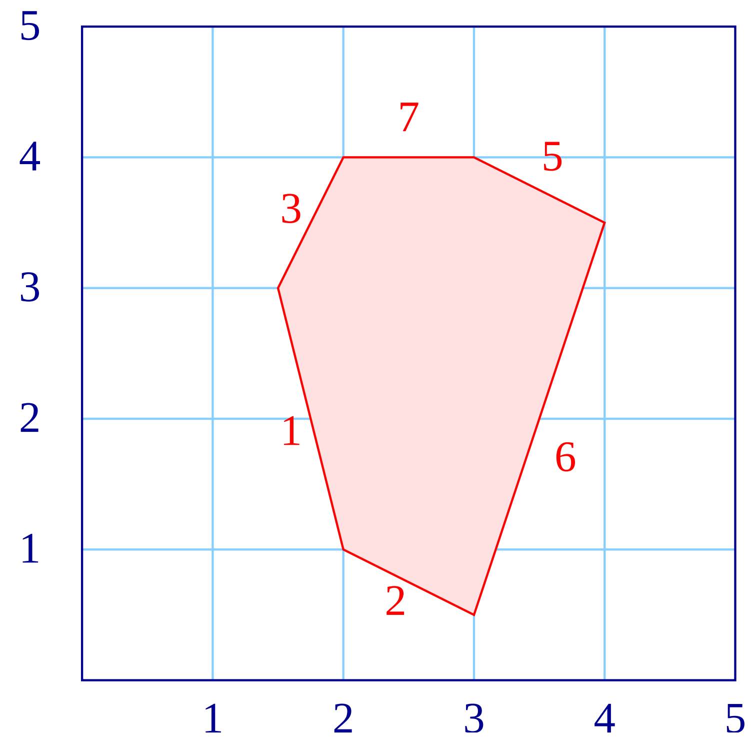
Idea: Fourier-Motzkin Elimination

- Successively remove individual variables x !
- All in-equations with positive occurrences of x yield lower bounds.
- All in-equations with negative occurrences of x yield upper bounds.
- All lower bounds must be at most as big as all upper bounds ;-))

Jean Baptiste Joseph Fourier,   1768–1830

Example:

$$
\begin{array}{lcll}
9 & \leq & 4x_1 + x_2 & (1) \\
4 & \leq & x_1 + 2x_2 & (2) \\
0 & \leq & 2x_1 - x_2 & (3) \\
6 & \leq & x_1 + 6x_2 & (4) \\
-11 & \leq & -x_1 - 2x_2 & (5) \\
-17 & \leq & -6x_1 + 2x_2 & (6) \\
-4 & \leq & -x_2 & (7)
\end{array}
$$



For x_1 we obtain:

$$\begin{array}{lll l} 9 & \leq & 4x_1 + x_2 & (1) \\ 4 & \leq & x_1 + 2x_2 & (2) \\ 0 & \leq & 2x_1 - x_2 & (3) \\ 6 & \leq & x_1 + 6x_2 & (4) \\ -11 & \leq & -x_1 - 2x_2 & (5) \\ -17 & \leq & -6x_1 + 2x_2 & (6) \\ -4 & \leq & -x_2 & (7) \end{array} \qquad \begin{array}{lll l} \frac{9}{4} - \frac{1}{4}x_2 & \leq & x_1 & (1) \\ 4 - 2x_2 & \leq & x_1 & (2) \\ \frac{1}{2}x_2 & \leq & x_1 & (3) \\ 6 - 6x_2 & \leq & x_1 & (4) \\ x_1 & \leq & 11 - 2x_2 & (5) \\ x_1 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (6) \\ -4 & \leq & -x_2 & (7) \end{array}$$

If such an x_1 exists, all lower bounds must be bounded by all upper bounds, i.e.,

$$
\begin{array}{rclcrcl}
\frac{9}{4} - \frac{1}{4}x_2 & \leq & 11 - 2x_2 & (1,5) & & -35 & \leq & -7x_2 & (1,5) \\
\frac{9}{4} - \frac{1}{4}x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (1,6) & & -\frac{7}{12} & \leq & \frac{7}{12}x_2 & (1,6) \\
4 - 2x_2 & \leq & 11 - 2x_2 & (2,5) & & -7 & \leq & 0 & (2,5) \\
4 - 2x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (2,6) & & \frac{7}{6} & \leq & \frac{7}{3}x_2 & (2,6) \\
\frac{1}{2}x_2 & \leq & 11 - 2x_2 & (3,5) & \text{or} & -22 & \leq & -5x_2 & (3,5) \\
\frac{1}{2}x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (3,6) & & -\frac{17}{6} & \leq & -\frac{1}{6}x_2 & (3,6) \\
6 - 6x_2 & \leq & 11 - 2x_2 & (4,5) & & -5 & \leq & 4x_2 & (4,5) \\
6 - 6x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (4,6) & & \frac{19}{6} & \leq & \frac{19}{3}x_2 & (4,6) \\
-4 & \leq & -x_2 & (7) & & -4 & \leq & -x_2 & (7)
\end{array}
$$

$$
\begin{array}{rcll}
\frac{9}{4} - \frac{1}{4}x_2 & \leq & 11 - 2x_2 & (1,5) \\
\frac{9}{4} - \frac{1}{4}x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (1,6) \\
4 - 2x_2 & \leq & 11 - 2x_2 & (2,5) \\
4 - 2x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (2,6) \\
\frac{1}{2}x_2 & \leq & 11 - 2x_2 & (3,5) \\
\frac{1}{2}x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (3,6) \\
6 - 6x_2 & \leq & 11 - 2x_2 & (4,5) \\
6 - 6x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (4,6) \\
-4 & \leq & -x_2 & (7)
\end{array}
\quad \text{or} \quad
\begin{array}{rcll}
-5 & \leq & -x_2 & (1,5) \\
-1 & \leq & x_2 & (1,6) \\
-7 & \leq & 0 & (2,5) \\
\frac{1}{2} & \leq & x_2 & (2,6) \\
-\frac{22}{5} & \leq & -x_2 & (3,5) \\
-17 & \leq & -x_2 & (3,6) \\
-\frac{5}{4} & \leq & x_2 & (4,5) \\
\frac{1}{2} & \leq & x_2 & (4,6) \\
-4 & \leq & -x_2 & (7)
\end{array}
$$

This is the one-variable case which we can solve exactly:

$$\max\{-1, \boxed{\tfrac{1}{2}}, -\tfrac{5}{4}, \tfrac{1}{2}\} \quad \leq \quad x_2 \quad \leq \quad \min\{5, \tfrac{22}{5}, 17, \boxed{4}\}$$

From which we conclude: $x_2 \in [\frac{1}{2}, 4]$:-)

In General:

- The original system has a solution over $\mathbb{Q}$ iff the system after elimination of one variable has a solution over $\mathbb{Q}$:-)
- Every elimination step may square the number of in-equations $\Longrightarrow$ exponential run-time :-((
- It can be modified such that it also decides satisfiability over $\mathbb{Z}$ $\Longrightarrow$ Omega Test

William Worthington Pugh, Jr.
University of Maryland, College Park

Idea:

- We successively remove variables. Thereby we omit division ...
- If x only occurs with coeffient ± 1, we apply Fourier-Motzkin elimination :-)
- Otherwise, we provide a bound for a positive multiple of x ...

Consider, e.g., (1) and (6) :

$$\begin{array}{rcl} 6 \cdot x_1 & \leq & 17 + 2x_2 \\ 9 - x_2 & \leq & 4 \cdot x_1 \end{array}$$

W.l.o.g., we only consider strict in-equations:

$$\begin{aligned} 6 \cdot x_1 &< 18 + 2x_2 \\ 8 - x_2 &< 4 \cdot x_1 \end{aligned}$$

... where we always divide by gcds:

$$\begin{aligned} 3 \cdot x_1 &< 9 + x_2 \\ 8 - x_2 &< 4 \cdot x_1 \end{aligned}$$

This implies:

$$3 \cdot (8 - x_2) < 4 \cdot (9 + x_2)$$

We thereby obtain:

- If one derived in-equation is unsatisfiable, then also the overall system :-)
- If all derived in-equations are satisfiable, then there is a solution which, however, need not be integer :-(
- An integer solution is guaranteed to exist if there is sufficient separation between lower and upper bound ...
- Assume $\alpha < a \cdot x \qquad b \cdot x < \beta$.

  Then it should hold that:

$$b \cdot \alpha < a \cdot \beta$$

  and moreover:

$$\boxed{a \cdot b} < a \cdot \beta - b \cdot \alpha$$

... in the Example:

$$12 \quad < \quad 4 \cdot (9 + x_2) - 3 \cdot (8 - x_2)$$

or:

$$12 \quad < \quad 12 + 7x_2$$

or:

$$0 \quad < \quad x_2$$

In the example, also these strengthened in-equations are satisfiable

$\Longrightarrow$ the system has a solution over $\mathbb{Z}$:-)

Discussion:

- If the strengthened in-equations are satisfiable, then also the original system. The reverse implication may be wrong :-(
- In the case where upper and lower bound are not sufficiently separated, we have:

$$a \cdot \beta \leq b \cdot \alpha + \boxed{a \cdot b}$$

or:

$$b \cdot \alpha < ab \cdot x < b \cdot \alpha + \boxed{a \cdot b}$$

Division with b yields:

$$\alpha < a \cdot x < \alpha + \boxed{a}$$

$\Longrightarrow$ $\boxed{\alpha + i = a \cdot x}$ for some $i \in \{1, \ldots, a-1\}$!!!

Discussion (cont.):

→ Fourier-Motzkin Elimination is not the best method for rational systems of in-equations.

→ The Omega test is necessarily exponential :-)

If the system is solvable, the test generally terminates rapidly.

It may have problems with unsolvable systems :-(

→ Also for ILP, there are other/smarter algorithms ...

→ For programming language problems, however, it seems to behave quite well :-)

4. Generalization to a Logic

Disjunction:

$$\begin{array}{l} (x - 2y = 15 \quad \wedge \quad x + y = 7) \qquad \vee \\ (x + y = 6 \qquad \wedge \quad 3x + z = -8) \end{array}$$

Quantors:

$$\exists\, x : \quad z - 2x = 42 \quad \wedge \quad z + x = 19$$

4. Generalization to a Logic

Disjunction:

$$
\begin{array}{llll}
(x - 2y = 15 & \wedge & x + y = 7) & \vee \\
(x + y = 6 & \wedge & 3x + z = -8) &
\end{array}
$$

Quantors:

$$\exists\, x : \quad z - 2x = 42 \quad \wedge \quad z + x = 19$$

$\Longrightarrow$ Presburger Arithmetic

Mojzesz Presburger, 1904–1943 (?)

Presburger Arithmetic $=$ full arithmetic without multiplication

Presburger Arithmetic $=$ full arithmetic without multiplication

Arithmetic : highly undecidable :-(

even incomplete :-((

Presburger Arithmetic = full arithmetic without multiplication

Arithmetic : highly undecidable :-(

even incomplete :-((

$\Longrightarrow$ Hilbert's 10th Problem

$\Longrightarrow$ Gödel's Theorem