$$
\begin{aligned}
[\![;]\!]\,(\rho,\mu) &= (\rho,\mu) & \\
\\
[\![\mathrm{Pos}\,(e)]\!]\,(\rho,\mu) &= (\rho,\mu) & \text{if } [\![e]\!]\,\rho \neq 0 \\
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// $[\![e]\!]$: evaluation of the expression e, e.g.

// $[\![x+y]\!]\,\{x \mapsto 7, y \mapsto -1\} = 6$

// $[\![!(x == 4)]\!]\,\{x \mapsto 5\} = 1$

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$$[\![R = e;]\!]\,(\rho,\mu) \quad = \quad (\boxed{\rho \oplus \{R \mapsto [\![e]\!]\,\rho\}}\,,\mu)$$

// where "$\oplus$" modifies a mapping at a given argument

$$\llbracket R = M[e]; \rrbracket\,(\rho, \mu) \quad = \quad (\boxed{\rho \oplus \{R \mapsto \mu(\llbracket e \rrbracket\,\rho))\}}, \mu)$$

$$\llbracket M[e_1] = e_2; \rrbracket\,(\rho, \mu) \quad = \quad (\rho, \boxed{\mu \oplus \{\llbracket e_1 \rrbracket\,\rho \mapsto \llbracket e_2 \rrbracket\,\rho\}})$$

Example:

$\llbracket x = x + 1; \rrbracket\,(\{x \mapsto 5\}, \mu) = (\rho, \mu)$ where:

$$\begin{aligned} \rho &= \{x \mapsto 5\} \oplus \{x \mapsto \llbracket x + 1 \rrbracket\,\{x \mapsto 5\}\} \\ &= \{x \mapsto 5\} \oplus \{x \mapsto 6\} \\ &= \{x \mapsto 6\} \end{aligned}$$

A path $\pi = k_1 k_2 \ldots k_m$ is a computation for the state s if:

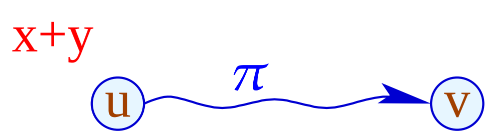
$$s \in def\,(\llbracket k_m \rrbracket \circ \ldots \circ \llbracket k_1 \rrbracket)$$

The result of the computation is:

$$\llbracket \pi \rrbracket\, s = (\llbracket k_m \rrbracket \circ \ldots \circ \llbracket k_1 \rrbracket)\, s$$

Application:

Assume that we have computed the value of red$x + y$ at program point u:



We perform a computation along path π and reach v where we evaluate again $x + y$ ...

Idea:

If x and y have not been modified in π, then evaluation of $x + y$ at v must return the same value as evaluation at u :-)

We can check this property at every edge in π :-}

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More generally:

Assume that the values of the expressions $A = \{e_1, \ldots, e_r\}$ are available at u.

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More generally:

Assume that the values of the expressions $A = \{e_1, \ldots, e_r\}$ are available at u.

Every edge k transforms this set into a set $[\![k]\!]^\sharp\, A$ of expressions whose values are available after execution of k ...

... which transformations can be composed to the effect of a path $\pi = k_1 \ldots k_r$:

$$[\![\pi]\!]^\sharp = [\![k_r]\!]^\sharp \circ \ldots \circ [\![k_1]\!]^\sharp$$

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The effect $[\![k]\!]^\sharp$ of an edge $k = (u, lab, v)$ only depends on the label *lab*, i.e., $[\![k]\!]^\sharp = [\![lab]\!]^\sharp$

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The effect $[\![k]\!]^\sharp$ of an edge $k = (u, lab, v)$ only depends on the label *lab*, i.e., $[\![k]\!]^\sharp = [\![lab]\!]^\sharp$ where:

$$\begin{array}{lcl} [\![;]\!]^\sharp\, A & = & A \\ [\![Pos(e)]\!]^\sharp\, A & = & [\![Neg(e)]\!]^\sharp\, A \quad = \quad A \cup \{e\} \\ [\![x = e;]\!]^\sharp\, A & = & (A \cup \{e\}) \backslash Expr_x \qquad \text{where} \\ & & Expr_x \text{ all expressions which contain } x \end{array}$$

$$
\begin{aligned}
[\![x = M[e];]\!]^\sharp\, A &= (A \cup \{e\}) \backslash Expr_x \\
[\![M[e_1] = e_2;]\!]^\sharp\, A &= A \cup \{e_1, e_2\}
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By that, every path can be analyzed :-)

A given program may admit several paths :-(

For any given input, another path may be chosen :-((

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For any given input, another path may be chosen :-((

$\Longrightarrow$ We require the set:

$$\mathcal{A}[v] \quad = \quad \bigcap \{ [\![\pi]\!]^\sharp \emptyset \mid \pi : start \to^* v \}$$

Concretely:

$\longrightarrow$ We consider all paths π which reach v.

$\longrightarrow$ For every path π, we determine the set of expressions which are available along π.

$\longrightarrow$ Initially at program start, nothing is available :-)

$\longrightarrow$ We compute the intersection $\Longrightarrow$ safe information

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- $\longrightarrow$ We consider all paths π which reach v.
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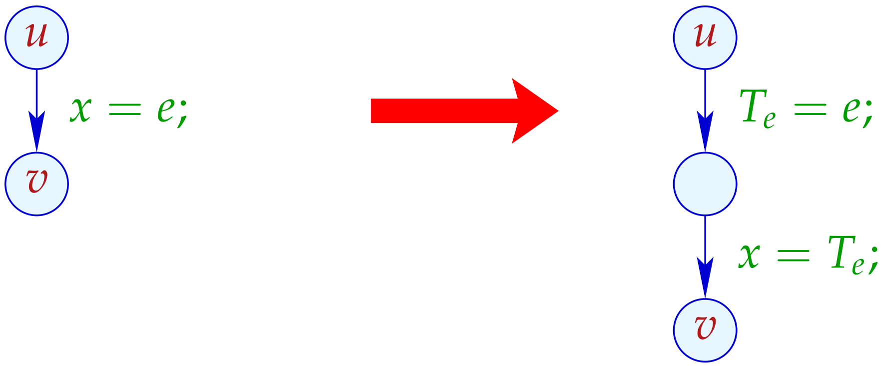
How do we exploit this information ???

Transformation 1.1:

We provide novel registers T_e as storage for the e:



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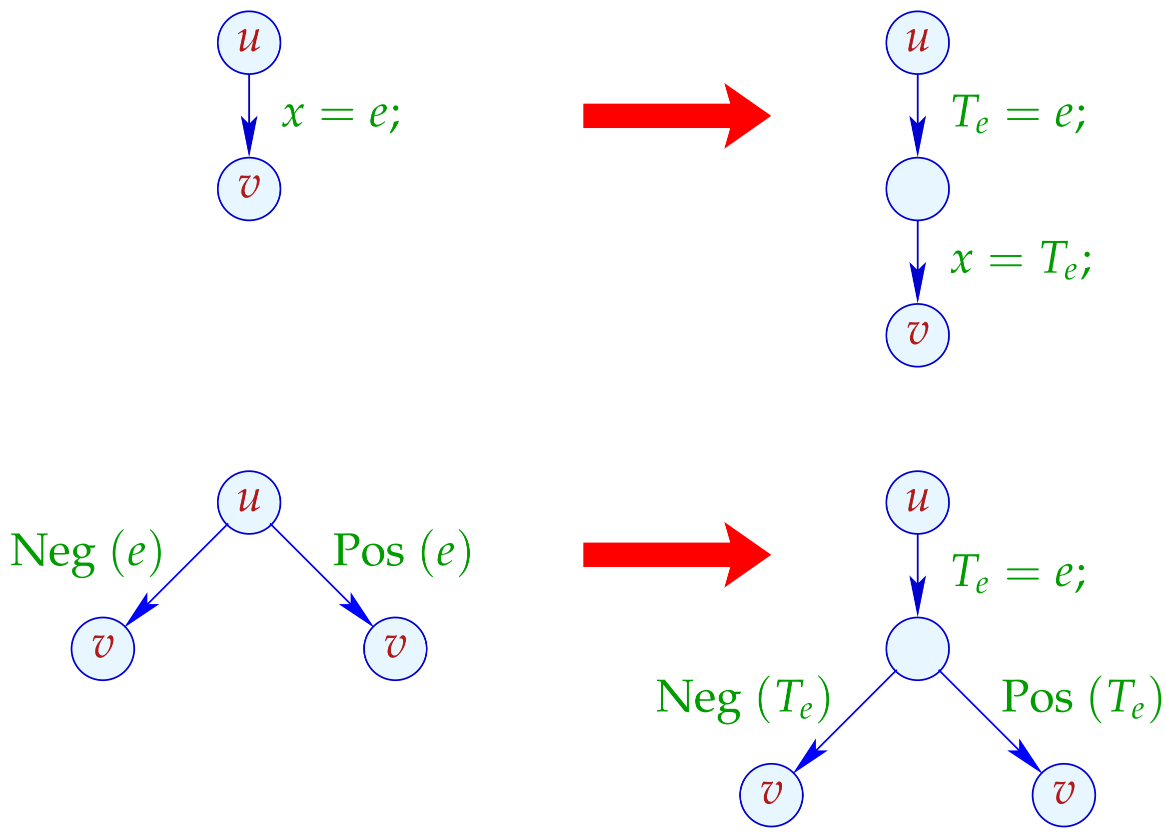
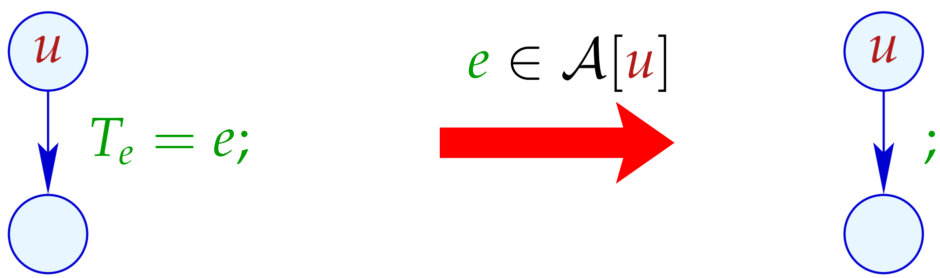


... analogously for $R = M[e];$ and $M[e_1] = e_2;$.

Transformation 1.2:

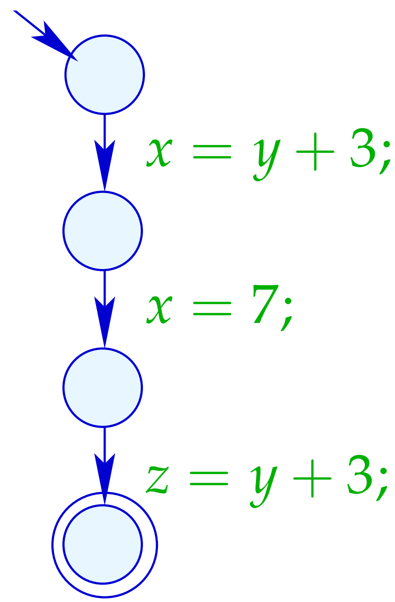
If e is available at program point u, then e need not be re-evaluated:



We replace the assignment with Nop :-)

Example:

$$
\begin{aligned}
x &= y + 3; \\
x &= 7; \\
z &= y + 3;
\end{aligned}
$$



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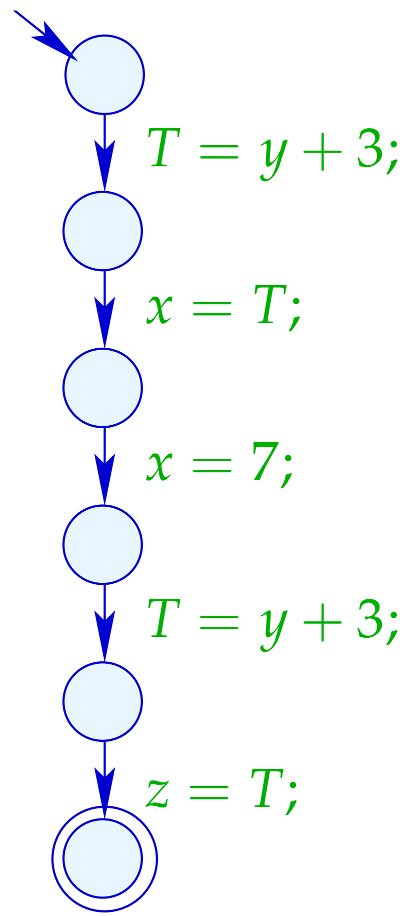

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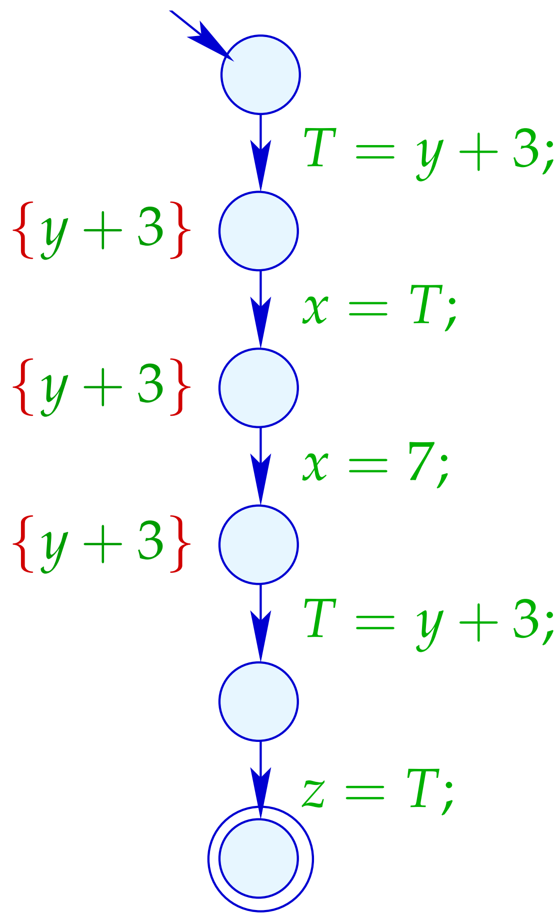

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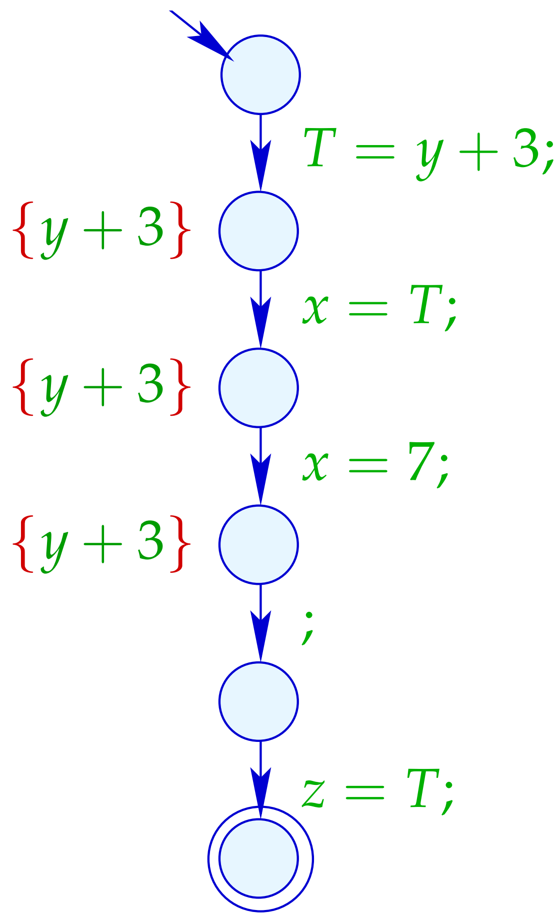
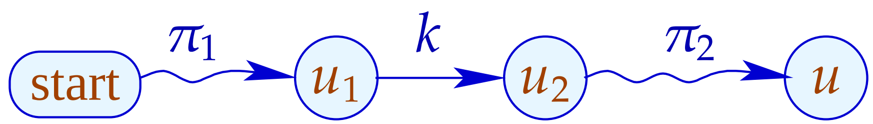


Correctness: (Idea)

Transformation 1.1 preserves the semantics and $\mathcal{A}[u]$ for all program points u :-)

Assume $\pi : start \to^* u$ is the path taken by a computation.
If $e \in \mathcal{A}[u]$, then also $e \in [\![\pi]\!]^\sharp \emptyset$.

Therefore, π can be decomposed into:



with the following properties:

- The expression e is evaluated at the edge k;
- The expression e is not removed from the set of available expressions at any edge in π_2, i.e., no variable of e receives a new value :-)

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$$\Longrightarrow$$

The register T_e contains the value of e whenever u is reached :-))

Warning:

Transformation 1.1 is only meaningful for assignments $x = e;$ where:

- $\rightarrow$ $x \notin Vars(e);$
- $\rightarrow$ $e \notin Vars;$
- $\rightarrow$ the evaluation of e is non-trivial :-}

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Which leaves us with the following question ...

Question:

How do we compute $\mathcal{A}[u]$ for every program point u ??

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How can we compute $\mathcal{A}[u]$ for every program point u ??

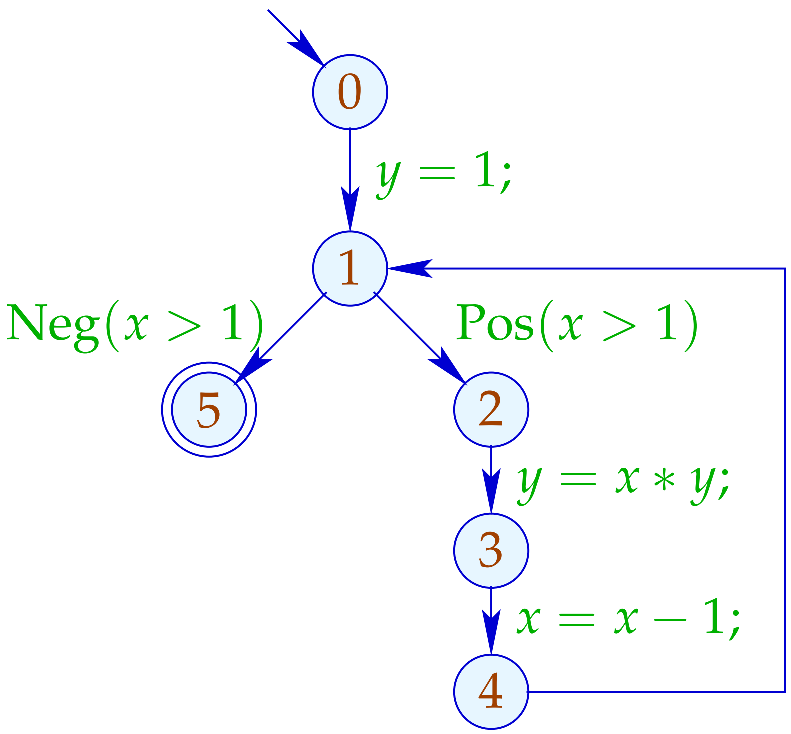
We collect all restrictions to the values of $\mathcal{A}[u]$ into a system of constraints:

$$
\begin{array}{lcll}
\mathcal{A}[start] & \subseteq & \emptyset & \\
\mathcal{A}[v] & \subseteq & [\![k]\!]^\sharp\,(\mathcal{A}[u]) & \qquad k = (u, \_, v) \quad \text{edge}
\end{array}
$$

Wanted:

- a maximally large solution (??)
- an algorithm which computes this :-)

Example:



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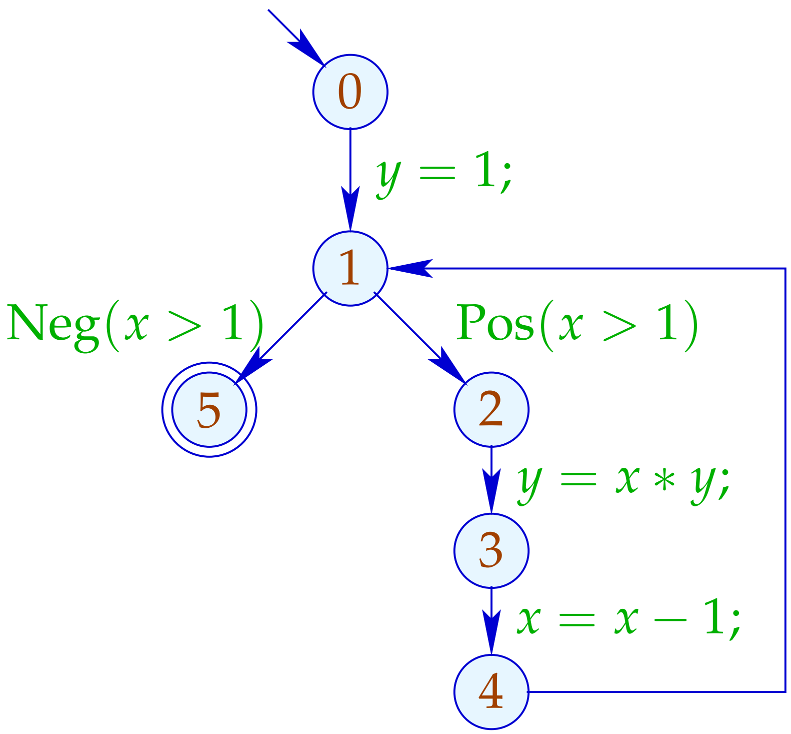

$$\mathcal{A}[0] \quad \subseteq \quad \emptyset$$

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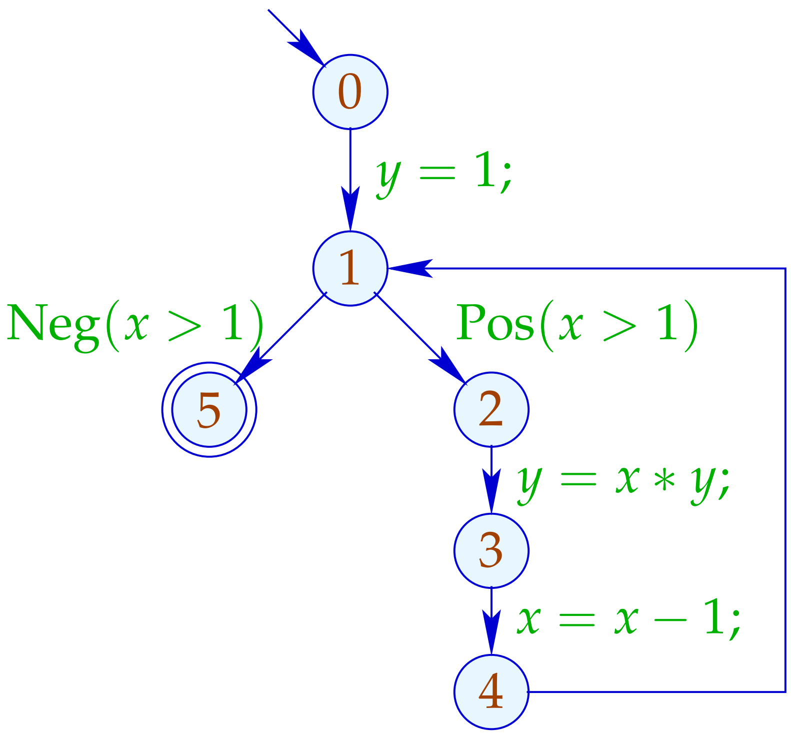

$$\begin{aligned} \mathcal{A}[0] &\subseteq \emptyset \\ \mathcal{A}[1] &\subseteq (\mathcal{A}[0] \cup \{1\}) \backslash Expr_y \\ \mathcal{A}[1] &\subseteq \mathcal{A}[4] \end{aligned}$$

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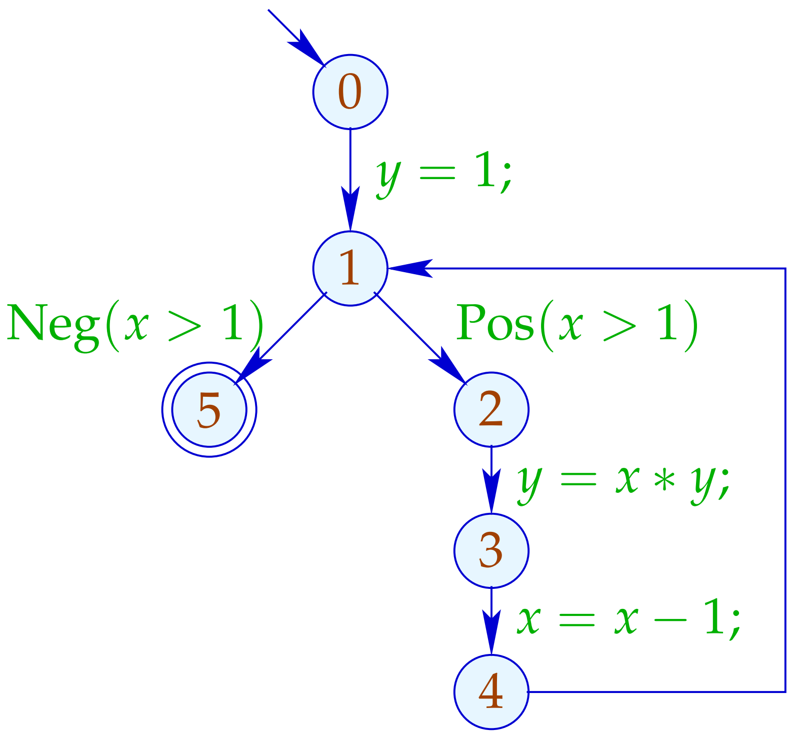

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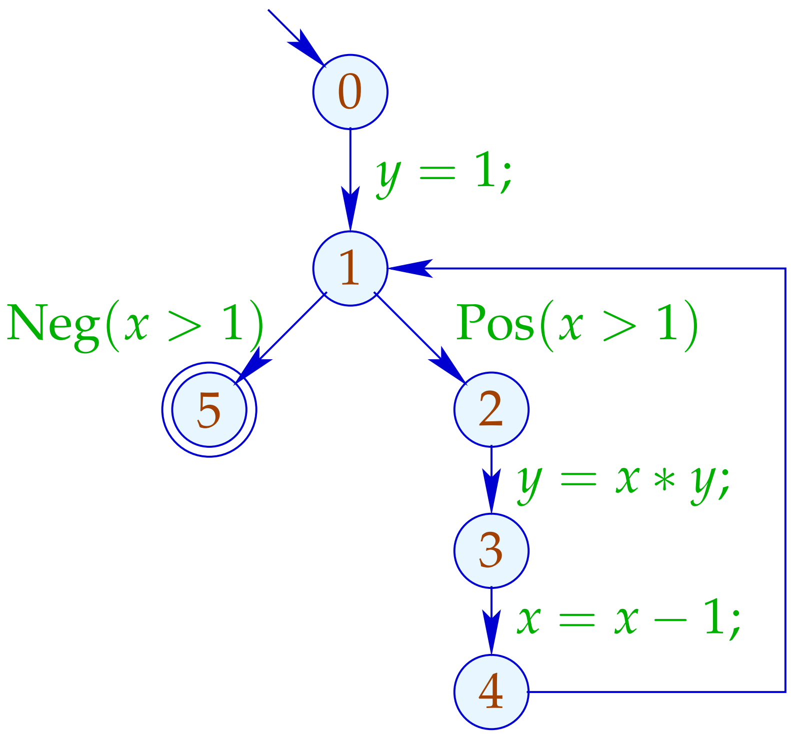

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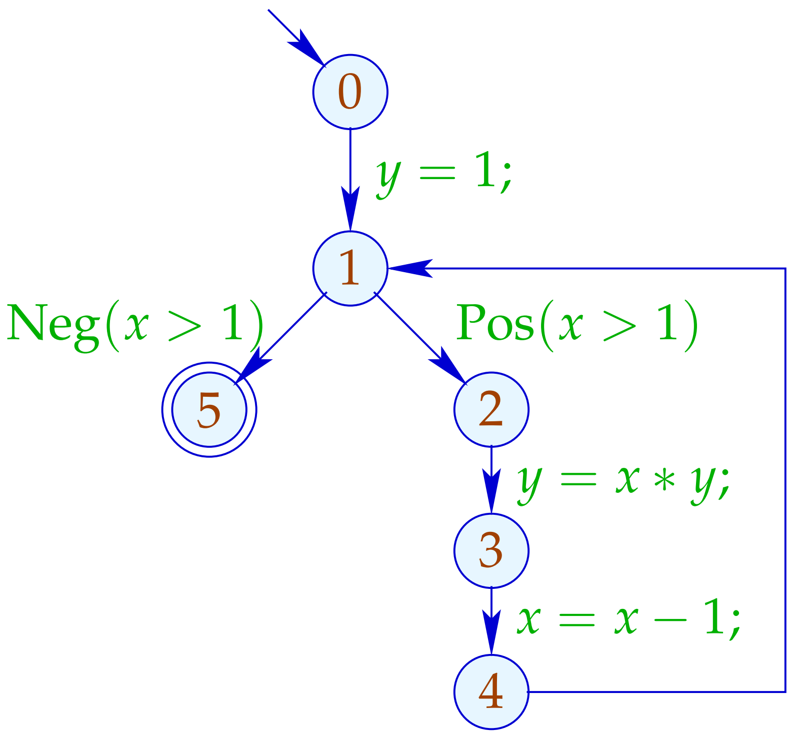

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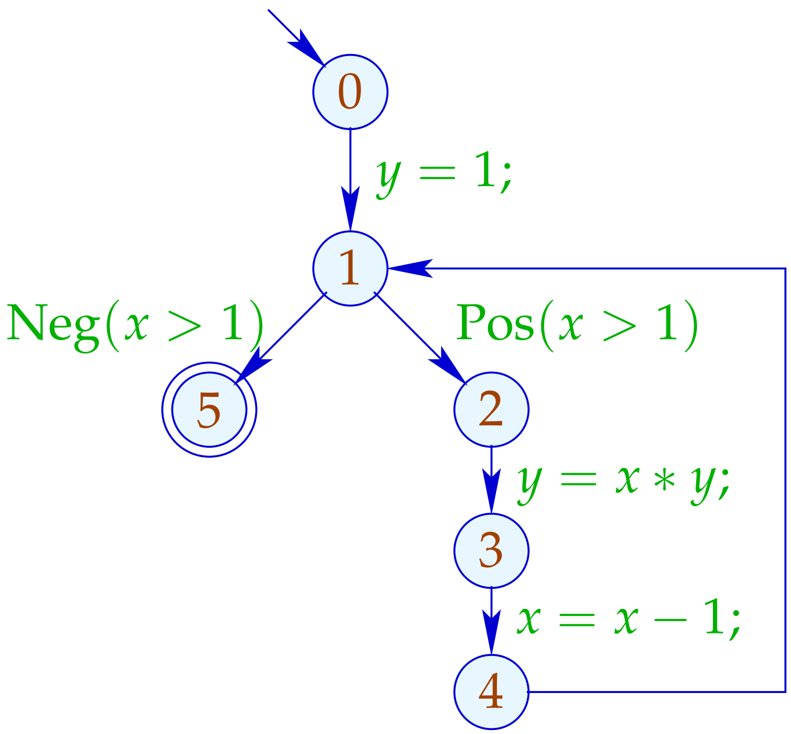

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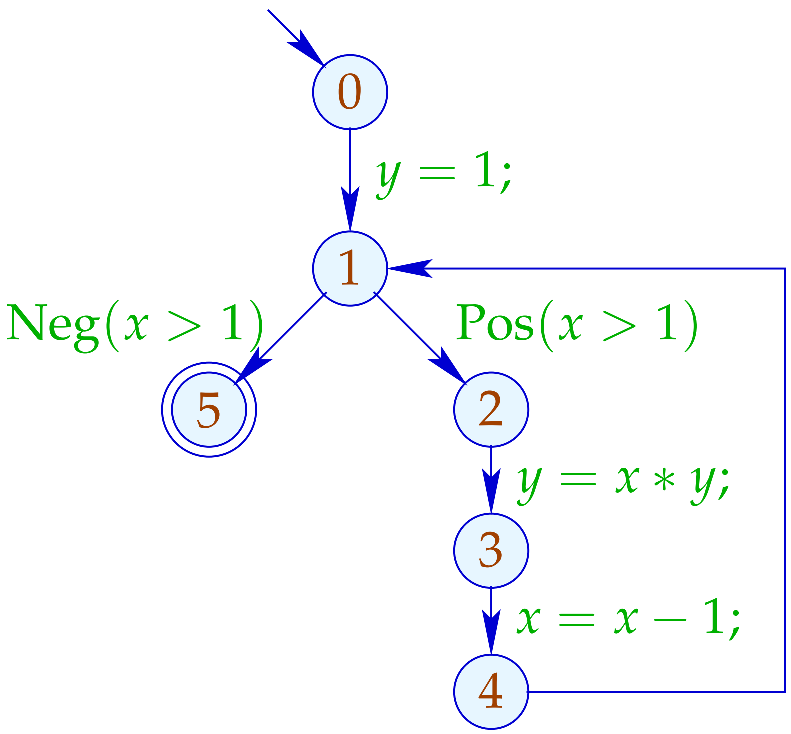

Solution:

$$
\begin{array}{lll}
\mathcal{A}[0] & = & \emptyset \\
\mathcal{A}[1] & = & \{1\} \\
\mathcal{A}[2] & = & \{1, x > 1\} \\
\mathcal{A}[3] & = & \{1, x > 1\} \\
\mathcal{A}[4] & = & \{1\} \\
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\end{array}
$$

Observation:

- The possible values for $\mathcal{A}[u]$ form a complete lattice:

$$\mathbb{D} = 2^{Expr} \quad \text{with} \quad B_1 \sqsubseteq B_2 \quad \text{iff} \quad B_1 \supseteq B_2$$

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- The functions $[\![k]\!]^\sharp : \mathbb{D} \to \mathbb{D}$ are monotonic, i.e.,

$$[\![k]\!]^\sharp(B_1) \sqsubseteq [\![k]\!]^\sharp(B_2) \quad \text{iff} \quad B_1 \sqsubseteq B_2$$

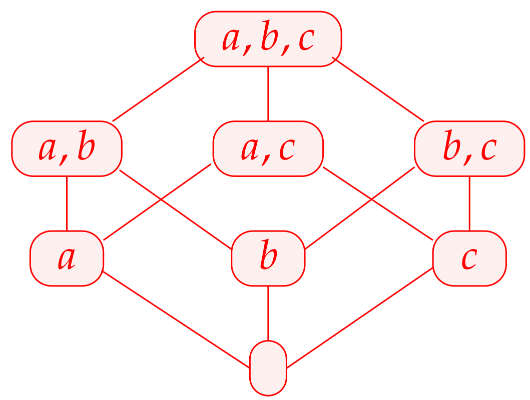
Background 2: complete Lattices

A set $\mathbb{D}$ together with a relation $\sqsubseteq \subseteq \mathbb{D} \times \mathbb{D}$ is a partial order if for all $a, b, c \in \mathbb{D}$,

$$a \sqsubseteq a \qquad \textit{reflexivity}$$
$$a \sqsubseteq b \wedge b \sqsubseteq a \implies a = b \qquad \textit{anti-symmetry}$$
$$a \sqsubseteq b \wedge b \sqsubseteq c \implies a \sqsubseteq c \qquad \textit{transitivity}$$

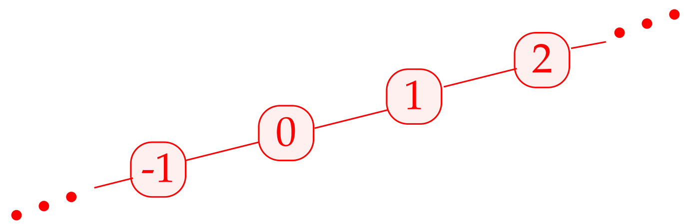
Examples:

1. $\mathbb{D} = 2^{\{a,b,c\}}$ with the relation "$\subseteq$" :

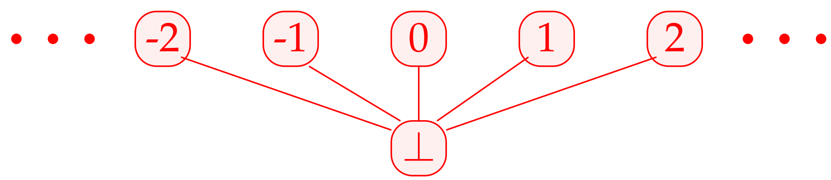


3. $\mathbb{Z}$ with the relation "$=$" :

3. $\mathbb{Z}$ with the relation "$\leq$" :



4. $\mathbb{Z}_{\perp} = \mathbb{Z} \cup \{\perp\}$ with the ordering:



$d \in \mathbb{D}$ is called upper bound for $X \subseteq \mathbb{D}$ if

$$x \sqsubseteq d \qquad \text{for all } x \in X$$

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2. $d \sqsubseteq y$ for every upper bound y of X.

Warning:

- $\{0, 2, 4, \ldots\} \subseteq \mathbb{Z}$ has no upper bound!
- $\{0, 2, 4\} \subseteq \mathbb{Z}$ has the upper bounds $4, 5, 6, \ldots$