4 Optimization of Functional Programs

Example:

$$\begin{aligned} \textbf{let rec } \text{fac } x \quad = & \quad \textbf{if } x \leq 1 \textbf{ then } 1 \\ & \quad \textbf{else } x \cdot \text{fac } (x-1) \end{aligned}$$

- There are no basic blocks :-(
- There are no loops :-(
- Virtually all functions are recursive :-((

Strategies for Optimization:

⟹ Improve specific inefficiencies such as:

- Pattern matching
- Lazy evaluation (if supported ;-)
- Indirections — Unboxing / Escape Analysis
- Intermediate data-structures — Deforestation

⟹ Detect and/or generate loops with basic blocks :-)

- Tail recursion
- Inlining
- **let**-Floating

Then apply general optimization techniques

... e.g., by translation into C ;-)

Warning:

Novel analysis techniques are needed to collect information about functional programs.

Example: Inlining

$$
\begin{array}{lcl}
\textbf{let}\ \ \mathsf{max}\ (x, y) & = & \textbf{if}\ \ x > y\ \ \textbf{then}\ \ x \\
 & & \textbf{else}\ \ y \\
\textbf{let}\ \ \mathsf{abs}\ z & = & \mathsf{max}\ (z, -z)
\end{array}
$$

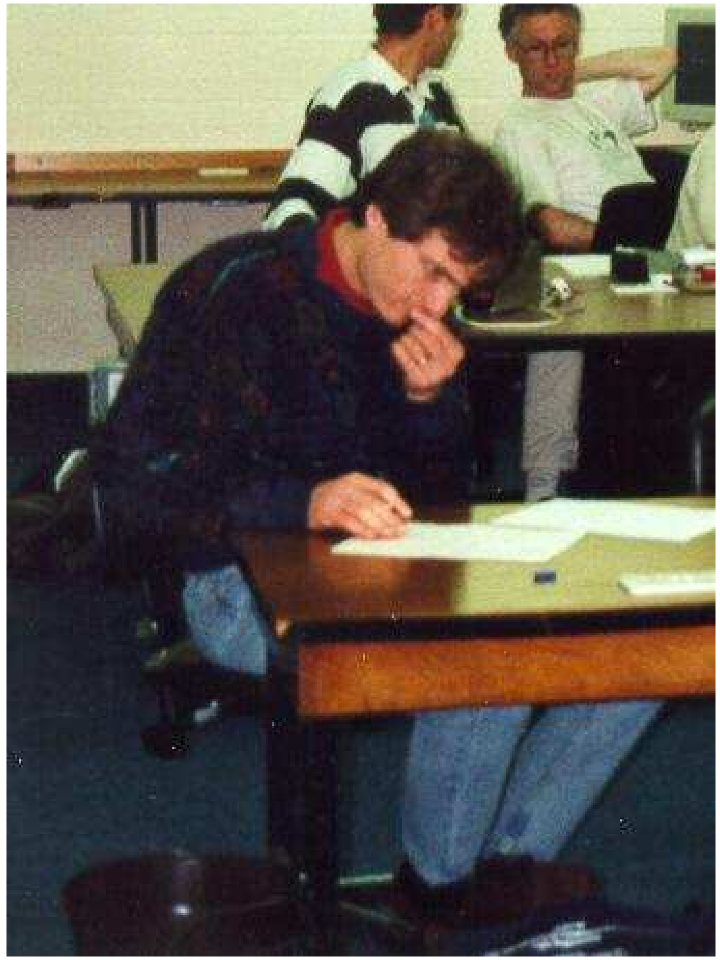
As result of the optimization we expect ...

$$
\begin{array}{lcl}
\textbf{let } \textsf{max}\ (x, y) & = & \textbf{if } x > y \textbf{ then } x \\
 & & \textbf{else } y \\
\textbf{let } \textsf{abs}\ z & = & \textbf{let } \quad x = z \\
 & & \textbf{and } \quad y = -z \\
 & & \textbf{in } \quad \boxed{\begin{array}{l}\textbf{if } x > y \textbf{ then } x \\ \textbf{else } y\end{array}} \\
 & & \textbf{end}
\end{array}
$$

Discussion:

For the beginning, max is just a name. We must find out which value it takes at run-time

$\Longrightarrow$ Value Analysis required !!



Nevin Heintze in the Australian team
of the Prolog-Programming-Contest, 1998

The complete picture:

4.1 A Simple Functional Language

For simplicity, we consider:

$$
\begin{array}{lcl}
e & ::= & b \mid (e_1, \ldots, e_k) \mid c\; e_1\; \ldots\; e_k \mid \textbf{fun}\; x \rightarrow e \\
 & & \mid (e_1\, e_2) \mid (\Box_1\; e) \mid (e_1\, \Box_2\, e_2) \mid \\
 & & \textbf{let}\; x_1 = e_1\; \textbf{and} \ldots \textbf{and}\; x_k = e_k\; \textbf{in}\; e_0 \mid \\
 & & \textbf{match}\; e_0\; \textbf{with}\; p_1 \rightarrow e_1 \mid \ldots \mid p_k \rightarrow e_k \\
p & ::= & b \mid x \mid c\; x_1 \ldots x_k \mid (x_1, \ldots, x_k) \\
t & ::= & \textbf{let rec}\; x_1 = e_1\; \textbf{and} \ldots \textbf{and}\; x_k = e_k\; \textbf{in}\; e
\end{array}
$$

where b is a constant, x is a variable, c is a (data-)constructor and $\Box_i$ are i-ary operators.

Discussion:

- **let rec** only occurs on top-level.
- Constructors and functions are always unary.
  Instead, there are explicit tuples :-)
- **if**-expressions and case distinction in function definitions is reduced to **match**-expressions.
- In case distinctions, we allow just simple patterns.

  $\Longrightarrow$ Complex patterns must be decomposed ...
- **let**-definitions correspond to basic blocks :-)
- Type-annotations at variables, patterns or expressions could provide further useful information
  — which we ignore :-)

... in the Example:

A definition of max may look as follows:

$$
\begin{array}{l}
\textbf{let}\ \mathsf{max}\ =\ \textbf{fun}\ x \rightarrow\ \ \textbf{match}\ x\ \textbf{with}\ (x_1, x_2)\ \rightarrow\ (\\
\qquad\qquad\qquad\qquad\qquad\qquad \textbf{match}\ x_1 < x_2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\ \ \textbf{with}\ \ \mathsf{True}\ :\ \ x_2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \mid\ \ \ \mathsf{False}\ :\ \ x_1 \\
\qquad\qquad\qquad\qquad\qquad\qquad)
\end{array}
$$

Accordingly, we have for abs :

$$\begin{array}{llll} \textbf{let}\ \mathsf{abs} & = & \textbf{fun}\ x \rightarrow & \textbf{let}\ z = (x, -x) \\ & & & \textbf{in}\ \mathsf{max}\ z \end{array}$$

4.2 A Simple Value Analysis

Idea:

For every subexpression e we collect the set $[\![e]\!]^\sharp$ of possiblevalues of e ...

Let V denote the set of occurring (classes of) constants, applications of constructors and functions. As our lattice, we choose:

$$\mathbb{V} = 2^V$$

As usual, we put up a constraint system:

- If e is a value, i.e., of the form: $b, c\,e, (e_1, \ldots, e_k)$, an operator application or $\textbf{fun}\, x \rightarrow e$ we generate the constraint:

$$[\![e]\!]^\sharp \supseteq \{e\}$$

- If $e \equiv (e_1\; e_2)$ and $f \equiv \textbf{fun}\, x \rightarrow e'$, then

$$\begin{array}{lll} [\![e]\!]^\sharp & \supseteq & (f \in [\![e_1]\!]^\sharp) \,?\, [\![e']\!]^\sharp : \emptyset \\ [\![x]\!]^\sharp & \supseteq & (f \in [\![e_1]\!]^\sharp) \,?\, [\![e_2]\!]^\sharp : \emptyset \end{array}$$

...

- int-values returned by operators are described by the unevaluated expression;

  Operator applications which return Boolean values, e.g., by $\{\mathsf{True}, \mathsf{False}\}$:-)

- If $e \equiv \mathbf{let}\ x_1 = e_1\ \mathbf{and} \ldots \mathbf{and}\ x_k = e_k\ \mathbf{in}\ e_0$, then we generate:

$$\begin{array}{lll} [\![x_i]\!]^\sharp & \supseteq & [\![e_i]\!]^\sharp \\ [\![e]\!]^\sharp & \supseteq & [\![e_0]\!]^\sharp \end{array}$$

- Assume $e \equiv \textbf{match } e_0 \textbf{ with } p_1 \ \rightarrow\ e_1 \mid \ldots \mid p_k \ \rightarrow\ e_k$.
  Then we generate for $p_i \equiv b$,

$$[\![e]\!]^\sharp \supseteq (b \in [\![e]\!]^\sharp)\,?\,[\![e_i]\!]^\sharp\ :\ \emptyset$$

If $p_i \equiv c\,y$ and $v \equiv c\,e'$ is a value, then

$$\begin{array}{lll}[\![e]\!]^\sharp & \supseteq & (v \in [\![e_0]\!]^\sharp)\,?\,[\![e_i]\!]^\sharp\ :\ \emptyset \\ [\![y]\!]^\sharp & \supseteq & (v \in [\![e_0]\!]^\sharp)\,?\,[\![e']\!]^\sharp\ :\ \emptyset\end{array}$$

If $p_i \equiv (y_1, \ldots, y_k)$ and $v \equiv (e'_1, \ldots, e'_k)$ is a value, then

$$\begin{array}{lll}[\![e]\!]^\sharp & \supseteq & (v \in [\![e_0]\!]^\sharp)\,?\,[\![e_i]\!]^\sharp\ :\ \emptyset \\ [\![y_j]\!]^\sharp & \supseteq & (v \in [\![e_0]\!]^\sharp)\,?\,[\![e'_j]\!]^\sharp\ :\ \emptyset\end{array}$$

If $p_i \equiv y$, then

$$\begin{array}{lll}[\![e]\!]^\sharp & \supseteq & [\![e_i]\!]^\sharp \\ [\![y]\!]^\sharp & \supseteq & [\![e_0]\!]^\sharp\end{array}$$

Example — The append-Function

Consider the concatenation of two lists. In Ocaml, we would write:

$$
\begin{array}{llll}
\textbf{let rec } \mathsf{app} & = & \textbf{fun } x \rightarrow & \textbf{match } x \textbf{ with} \\
& & & \quad [\,] \quad \rightarrow \quad \textbf{fun } y \rightarrow y \\
& & & \mid h :: t \rightarrow \quad \textbf{fun } y \rightarrow h :: \mathsf{app}\ t\ y \\
\textbf{in } \mathsf{app}\ [1;2]\ [3] & & &
\end{array}
$$

The analysis then results in:

$$
\begin{array}{lll}
[\![\mathsf{app}]\!]^\sharp & = & \{\textbf{fun } x \rightarrow \textbf{match} \ldots\} \\
[\![x]\!]^\sharp & = & \{[1;2], [1], [\,]\} \\
[\![\textbf{match} \ldots]\!]^\sharp & = & \{\textbf{fun } y \rightarrow y, \textbf{fun } y \rightarrow x :: \mathsf{app} \ldots\} \\
[\![y]\!]^\sharp & = & \{[3]\} \\
\ldots & &
\end{array}
$$

$$\begin{array}{lcl}
\ldots & & \\
[\![h]\!]^\sharp & = & \{1,2\} \\
[\![t]\!]^\sharp & = & \{[2],[]\} \\
[\![\mathsf{app}\ t]\!]^\sharp & = & \\
[\![\mathsf{app}\ [1;2]]\!]^\sharp & = & \{\mathbf{fun}\ y \to y, \mathbf{fun}\ y \to x :: \mathsf{app} \ldots\} \\
[\![\mathsf{app}\ t\ y]\!]^\sharp & = & \\
[\![\mathsf{app}\ [1;2]\ [3]]\!]^\sharp & = & \{[3], h :: \mathsf{app} \ldots\}
\end{array}$$

Values $c\ e$ or $(e_1, \ldots, e_k)$ now are interpreted as recursive calls $c\ [\![e]\!]^\sharp$ or $([\![e_1]\!]^\sharp, \ldots, [\![e_k]\!]^\sharp)$, respectively.

$\Longrightarrow$ regular tree grammar

... in the Example:

We obtain for $A = [\![\text{app}\ t\ y]\!]^\sharp$:

$$\begin{array}{lcl} A & \to & [3] \quad | \quad [\![h]\!]^\sharp :: A \\ [\![h]\!]^\sharp & \to & 1 \quad | \quad 2 \end{array}$$

Let $\mathcal{L}(e)$ denote the set of terms derivable from $[\![e]\!]^\sharp$ w.r.t. the regular tree grammar. Thus, e.g.,

$$\begin{array}{lcl} \mathcal{L}(h) & = & \{1, 2\} \\ \mathcal{L}(\text{app}\ t\ y) & = & \{[a_1; \ldots, a_r; 3] \mid r \geq 0, a_i \in \{1, 2\}\} \end{array}$$

4.3 An Operational Semantics

Idea:

We construct a Big-Step operational semantics which evaluates expressions w.r.t. an environment :-)

Values are of the form:

$$v ::= b \mid c\, v \mid (v_1, \ldots, v_k) \mid (\textbf{fun}\; x \;\rightarrow\; e, \eta)$$

Examples for Values:

$$\begin{array}{l} c\, 1 \\ [1;2] = \,::\, 1\; (::\, 2\; [\,]) \\ (\textbf{fun}\; x \rightarrow x{::}y, \{y \mapsto [5]\}) \end{array}$$

Expressions are evaluated w.r.t. an environment $\eta : Vars \to Values$.

The Big-Step operational semantics provides rules to infer the value to which an expression is evaluated w.r.t. a given environment...

Values:

$$(b, \eta) \Longrightarrow b$$

$$(\mathbf{fun}\, x \;\to\; e, \eta) \Longrightarrow (\mathbf{fun}\, x \;\to\; e, \eta)$$

$$\frac{(e, \eta) \Longrightarrow v}{(c\, e, \eta) \Longrightarrow c\, v}$$

$$\frac{(e_1, \eta) \Longrightarrow v_1 \quad \ldots \quad (e_k, \eta) \Longrightarrow v_k}{((e_1, \ldots, e_k), \eta) \Longrightarrow (v_1, \ldots, v_k)}$$

Global Definition:

$$\frac{\begin{array}{l} \textbf{let rec } \ldots \; x = e \; \ldots \; \textbf{in } \ldots \\ (e, \emptyset) \Longrightarrow v \end{array}}{(x, \eta) \Longrightarrow v}$$

Function Application:

$$\frac{(e_1, \eta) \Longrightarrow (\textbf{fun } x \;\rightarrow\; e, \eta_1) \qquad (e_2, \eta) \Longrightarrow v_2 \qquad (e, \eta_1 \oplus \{x \mapsto v_2\}) \Longrightarrow v_3}{(e_1\ e_2, \eta) \Longrightarrow v_3}$$

Case Distinction 1:

$$\frac{(e, \eta) \Longrightarrow b \qquad (e_i, \eta) \Longrightarrow v_i}{(\textbf{match } e \textbf{ with } p_1 \;\rightarrow\; e_1 \mid \ldots \mid p_k \;\rightarrow\; e_k, \eta) \Longrightarrow v_i}$$

if $\quad p_i \equiv b \quad$ is the first pattern which matches $\quad b \quad$:-)

Case Distinction 2:

$$\frac{(e, \eta) \Longrightarrow c\, v \qquad (e_i, \eta \oplus \{z \mapsto v\}) \Longrightarrow v_i}{(\textbf{match}\ e\ \textbf{with}\ p_1 \ \rightarrow\ e_1 \mid \ldots \mid p_k \ \rightarrow\ e_k, \eta) \Longrightarrow v_i}$$

if $p_i \equiv c\ z$ is the first pattern which matches $c\,v$:-)

Case Distinction 3:

$$\frac{(e, \eta) \Longrightarrow (v_1, \ldots, v_k) \qquad (e_i, \eta \oplus \{y_1 \mapsto v_1, \ldots, y_1 \mapsto v_k\}) \Longrightarrow v_i}{(\textbf{match } e \textbf{ with } p_1 \to e_1 \mid \ldots \mid p_k \to e_k, \eta) \Longrightarrow v_i}$$

if $p_i \equiv (y_1, \ldots, y_k)$ is the first pattern which matches $(v_1, \ldots, v_k)$:-)

Case Distinction 4:

$$\frac{(e, \eta) \Longrightarrow v \qquad (e_i, \eta \oplus \{x \mapsto v\}) \Longrightarrow v_i}{(\textbf{match}\ e\ \textbf{with}\ p_1\ \rightarrow\ e_1 \mid \ldots \mid p_k\ \rightarrow\ e_k, \eta) \Longrightarrow v_i}$$

if $p_i \equiv x$ is the first pattern which matches v :-)

Local Definitions:

$$\frac{\begin{array}{l}(e_1, \eta) \Longrightarrow v_1 \\ (e_2, \eta \oplus \{x_1 \mapsto v_1\}) \Longrightarrow v_2 \\ \ldots \\ (e_k, \eta \oplus \{x_1 \mapsto v_1, \ldots, x_{k-1} \mapsto v_{k-1}\}) \Longrightarrow v_k \\ (e_0, \eta \oplus \{x_1 \mapsto v_1, \ldots, x_k \mapsto v_k\}) \Longrightarrow v_0\end{array}}{(\textbf{let } x_1 = e_1 \textbf{ and} \ldots \textbf{and } x_k = e_k \textbf{ in } e_0, \eta) \Longrightarrow v_0}$$

Correctness of the Analysis:

For every (e, η) occurring in a proof for the program, it should hold:

- If $\eta(x) = v$, then $[v] \in \mathcal{L}(x)$.
- If $(e, \eta) \Longrightarrow v$, then $[v] \in \mathcal{L}(e)$ ...

- where $[v]$ is the stripped expression corresponding to v, i.e., obtained by removing all environments.

Conclusion:

$\mathcal{L}(e)$ returns a superset of the values to which e is evaluated :-)