A complete lattice (cl) $\mathbb{D}$ is a partial ordering where every subset $X \subseteq \mathbb{D}$ has a least upper bound $\bigsqcup X \in \mathbb{D}$.

Note:

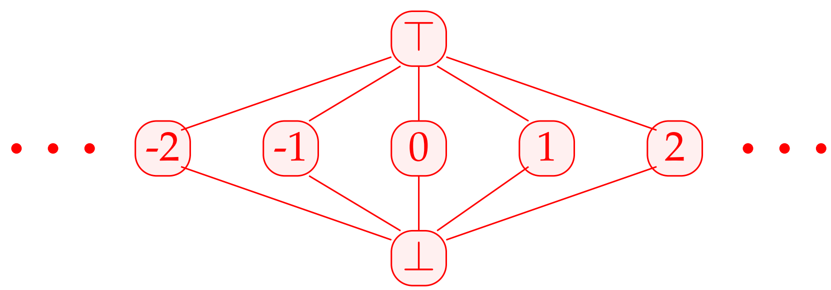
Every complete lattice has

→ a least element $\bot = \bigsqcup \emptyset \in \mathbb{D}$;

→ a greatest element $\top = \bigsqcup \mathbb{D} \in \mathbb{D}$.

Examples:

1. $\mathbb{D} = 2^{\{a,b,c\}}$ is a cl :-)
2. $\mathbb{D} = \mathbb{Z}$ with "$=$" is not.
3. $\mathbb{D} = \mathbb{Z}$ with "$\leq$" is neither.
4. $\mathbb{D} = \mathbb{Z}_\perp$ is also not :-(
5. With an extra element $\top$, we obtain the flat lattice $\mathbb{Z}_\perp^\top = \mathbb{Z} \cup \{\perp, \top\}$:



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Construct $U = \{u \in \mathbb{D} \mid \forall\, x \in X : \; u \sqsubseteq x\}$.

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Proof:

Construct $U = \{u \in \mathbb{D} \mid \forall\, x \in X :\; u \sqsubseteq x\}$.

// the set of all lower bounds of X :-)

Set: $g := \bigsqcup U$

Claim: $g = \bigsqcap X$

(1) g is a lower bound of X :

Assume $x \in X$. Then:

$u \sqsubseteq x$ for all $u \in U$

$\Longrightarrow$ x is an upper bound of U

$\Longrightarrow$ $g \sqsubseteq x$:-)

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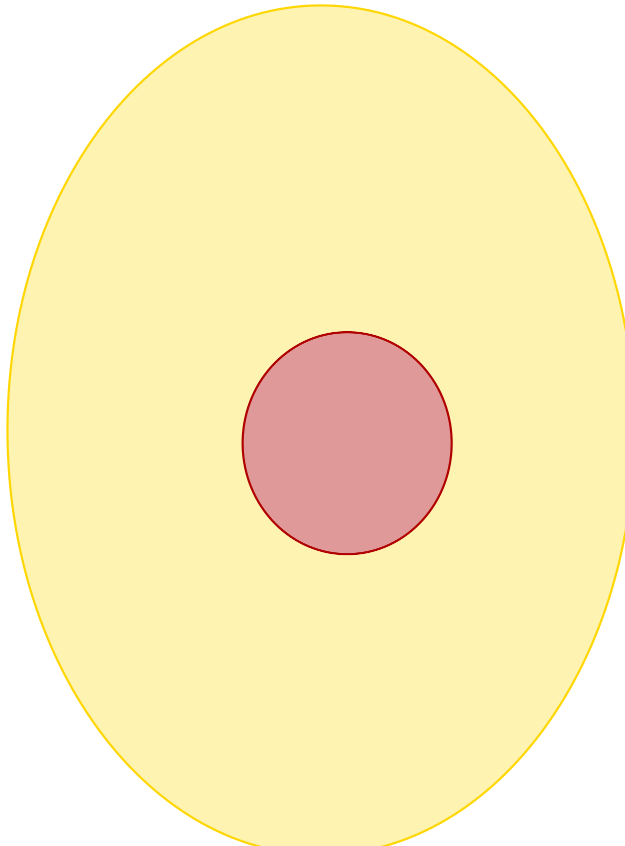

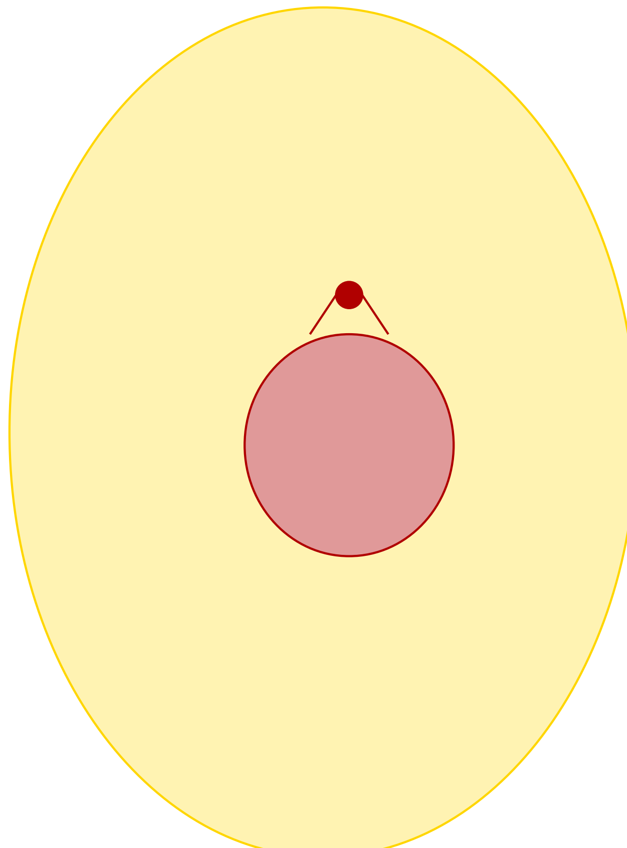

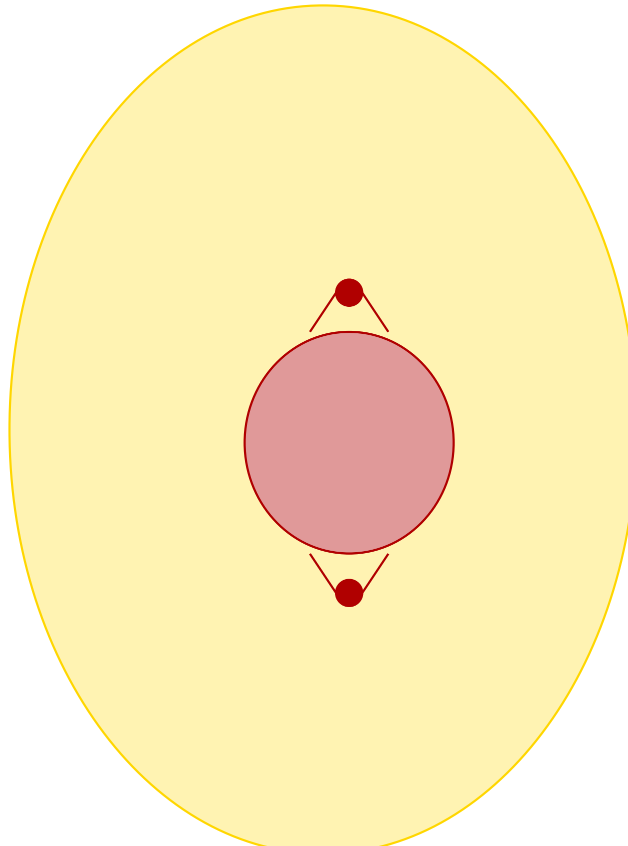

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where:

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Because:

$$x \sqsupseteq d_1 \wedge \ldots \wedge x \sqsupseteq d_k \quad \text{iff} \quad x \sqsupseteq \bigsqcup\{d_1, \ldots, d_k\} \qquad \text{:-)}$$

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- $\mathsf{dec}\,x = x - 1$ is monotonic.
- $\mathsf{inv}\,x = -x$ is not monotonic :-)

Theorem:

If $f_1 : \mathbb{D}_1 \to \mathbb{D}_2$ and $f_2 : \mathbb{D}_2 \to \mathbb{D}_3$ are monotonic, then also $f_2 \circ f_1 : \mathbb{D}_1 \to \mathbb{D}_3$:-)

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If $\mathbb{D}_2$ is a complete lattice, then the set $[\mathbb{D}_1 \to \mathbb{D}_2]$ of monotonic functions $f : \mathbb{D}_1 \to \mathbb{D}_2$ is also a complete lattice where

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In particular for $F \subseteq [\mathbb{D}_1 \to \mathbb{D}_2]$,

$$\bigsqcup F = f \quad \text{mit} \quad f\,x = \bigsqcup \{g\,x \mid g \in F\}$$

For functions $f_i\,x = a_i \cap x \cup b_i$, the operations "$\circ$", "$\sqcup$" and "$\sqcap$" can be explicitly defined by:

$$
\begin{array}{rcl}
(f_2 \circ f_1)\,x & = & \boxed{a_1 \cap a_2} \cap x \cup \boxed{a_2 \cap b_1 \cup b_2} \\
(f_1 \sqcup f_2)\,x & = & \boxed{(a_1 \cup a_2)} \cap x \cup \boxed{b_1 \cup b_2} \\
(f_1 \sqcap f_2)\,x & = & \boxed{(a_1 \cup b_1) \cap (a_2 \cup b_2)} \cap x \cup \boxed{b_1 \cap b_2}
\end{array}
$$

Wanted: minimally small solution for:

$$x_i \sqsupseteq f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n \qquad (*)$$

where all $f_i : \mathbb{D}^n \to \mathbb{D}$ are monotonic.

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Idea:

- Consider $F : \mathbb{D}^n \to \mathbb{D}^n$ where

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- If all f_i are monotonic, then also F :-)
- We succesively approximate a solution. We construct:

  $$\underline{\bot}, \quad F\,\underline{\bot}, \quad F^2\,\underline{\bot}, \quad F^3\,\underline{\bot}, \quad \ldots$$

Hope: We eventually reach a solution ... ???

Example: $\mathbb{D} = 2^{\{a,b,c\}}, \quad \sqsubseteq\ = \subseteq$

$$\begin{array}{lll} x_1 & \supseteq & \{a\} \cup x_3 \\ x_2 & \supseteq & x_3 \cap \{a, b\} \\ x_3 & \supseteq & x_1 \cup \{c\} \end{array}$$

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The Iteration:

	0	1	2	3	4
x_1	$\emptyset$				
x_2	$\emptyset$				
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Theorem

- $\underline{\perp}, F\,\underline{\perp}, F^2\,\underline{\perp}, \ldots$ form an ascending chain :

$$\underline{\perp} \quad \sqsubseteq \quad F\,\underline{\perp} \quad \sqsubseteq \quad F^2\,\underline{\perp} \quad \sqsubseteq \quad \ldots$$

- If $F^k\,\underline{\perp} = F^{k+1}\,\underline{\perp}$, a solution is obtained which is the least one :-)
- If all ascending chains are finite, such a k always exists.

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Proof

The first claim follows by complete induction:

Foundation: $F^0\,\underline{\perp} = \underline{\perp} \sqsubseteq F^1\,\underline{\perp}$:-)

Step: Assume $F^{i-1}\,\underline{\perp} \;\sqsubseteq\; F^{i}\,\underline{\perp}$. Then

$$F^{i}\,\underline{\perp} = F\,(F^{i-1}\,\underline{\perp}) \sqsubseteq F\,(F^{i}\,\underline{\perp}) = F^{i+1}\,\underline{\perp}$$

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Conclusion:

If $\mathbb{D}$ is finite, a solution can be found which is definitely the least :-)

Question:

What, if $\mathbb{D}$ is not finite ???

Theorem   Knaster – Tarski

Assume $\mathbb{D}$ is a complete lattice. Then every monotonic function $f : \mathbb{D} \to \mathbb{D}$ has a least fixpoint $d_0 \in \mathbb{D}$.

Let $P = \{d \in \mathbb{D} \mid f\, d \sqsubseteq d\}$.

Then $d_0 = \bigsqcap P$.

Bronisław Knaster (1893-1980), topology

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Proof:

(1) $d_0 \in P$:

$$f\, d_0 \sqsubseteq f\, d \sqsubseteq d \quad \text{for all } d \in P$$

$$\Longrightarrow \quad f\, d_0 \quad \text{is a lower bound of } P$$

$$\Longrightarrow \quad f\, d_0 \sqsubseteq d_0 \quad \text{since } d_0 = \bigsqcap P$$

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(2) $f\, d_0 = d_0$:

$$f\, d_0 \sqsubseteq d_0 \quad \text{by} \quad (1)$$

$\Longrightarrow$ $f(f\, d_0) \sqsubseteq f\, d_0$ by monotonicity of f

$\Longrightarrow$ $f\, d_0 \in P$

$\Longrightarrow$ $d_0 \sqsubseteq f\, d_0$ and the claim follows :-)

(3) d_0 is least fixpoint:

$$f\, d_1 = d_1 \sqsubseteq d_1 \quad \text{an other fixpoint}$$

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The least fixpoint d_0 is in P and a lower bound :-)

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Assume $x_i \sqsupseteq f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n$ $(*)$

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is a system of constraints where all $f_i : \mathbb{D}^n \to \mathbb{D}$ are monotonic.

$\Longrightarrow$ least solution of $(*)$ $==$ least fixpoint of F :-)

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Example 2: $\mathbb{D} = \mathbb{N} \cup \{\infty\}$

Assume $f\,x = x + 1$. Then

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$\Longrightarrow$ Ordinary iteration will never reach a fixpoint :-(

$\Longrightarrow$ Sometimes, transfinite iteration is needed :-)

Conclusion:

Systems of inequations can be solved through fixpoint iteration, i.e., by repeated evaluation of right-hand sides :-)

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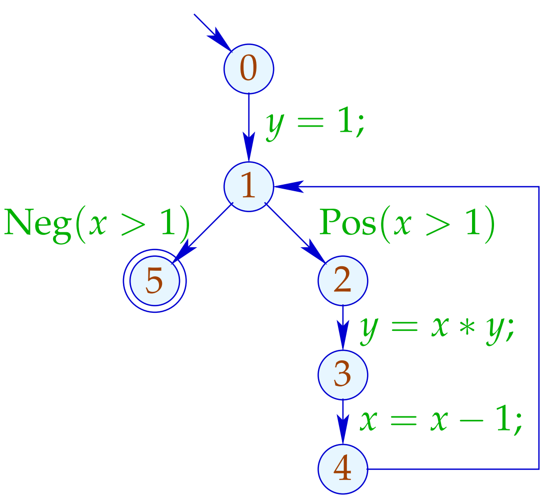

0
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2
3
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5

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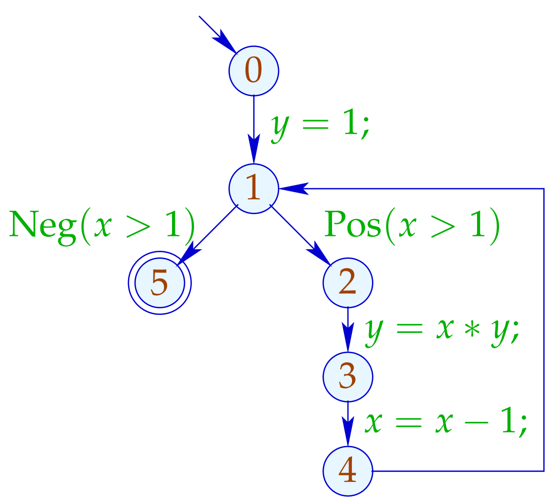

	1
0	$\emptyset$
1	$\{1, x > 1, x - 1\}$
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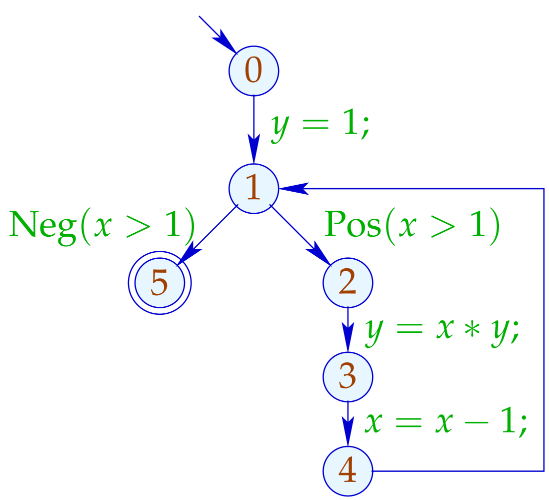

	1	2
0	$\emptyset$	$\emptyset$
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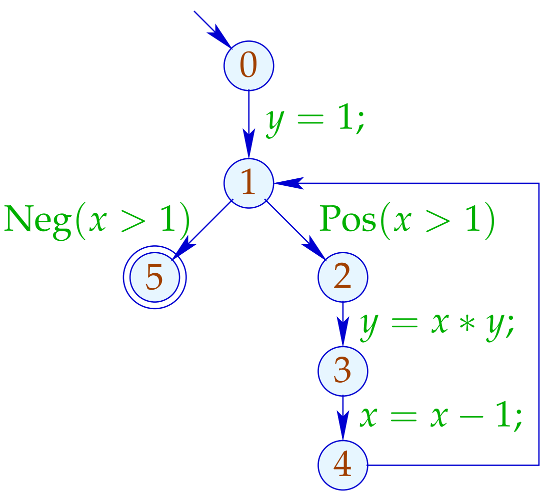

	1	2	3
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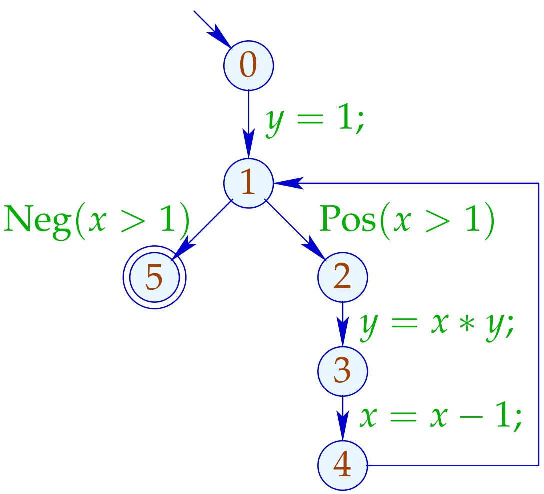


	1	2	3	4
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
1	$\{1, x > 1, x - 1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	*Expr*	$\{1, x > 1, x - 1\}$	$\{1, x > 1\}$	$\{1, x > 1\}$
3	$\{1, x > 1, x - 1\}$	$\{1, x > 1, x - 1\}$	$\{1, x > 1, x - 1\}$	$\{1, x > 1\}$
4	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
5	*Expr*	$\{1, x > 1, x - 1\}$	$\{1, x > 1\}$	$\{1, x > 1\}$

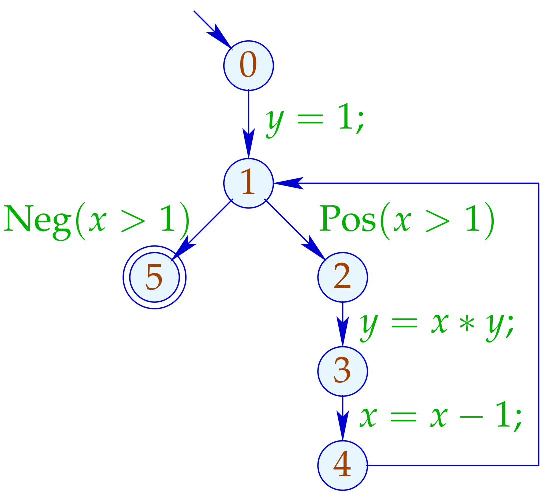
Conclusion:

Systems of inequations can be solved through fixpoint iteration, i.e., by repeated evaluation of right-hand sides :-)

Warning:

Naive fixpoint iteration is rather inefficient :-(

Example:



	1	2	3	4	5
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	
1	$\{1, x > 1, x - 1\}$	$\{1\}$	$\{1\}$	$\{1\}$	
2	*Expr*	$\{1, x > 1, x - 1\}$	$\{1, x > 1\}$	$\{1, x > 1\}$	
3	$\{1, x > 1, x - 1\}$	$\{1, x > 1, x - 1\}$	$\{1, x > 1, x - 1\}$	$\{1, x > 1\}$	dito
4	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	
5	*Expr*	$\{1, x > 1, x - 1\}$	$\{1, x > 1\}$	$\{1, x > 1\}$	

Idea: Round Robin Iteration

Instead of accessing the values of the last iteration, always use the current values of unknowns :-)

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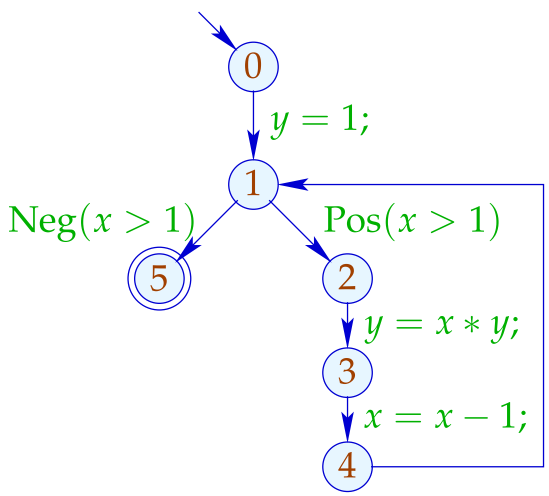
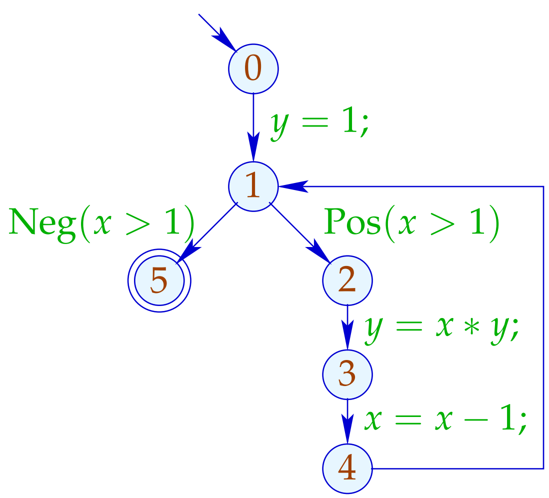


0
1
2
3
4
5

Idea: Round Robin Iteration

Instead of accessing the values of the last iteration, always use the current values of unknowns :-)

Example:

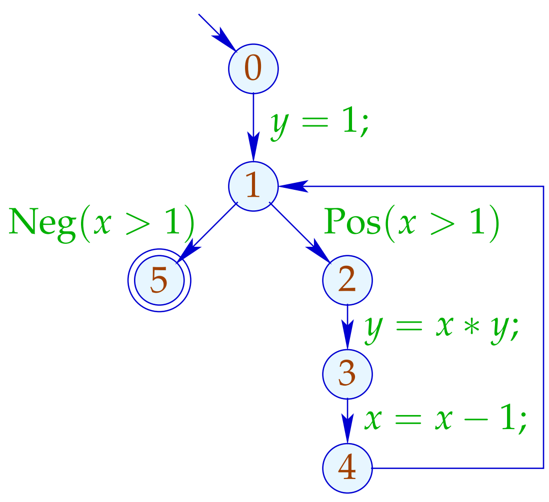


	1
0	$\emptyset$
1	$\{1\}$
2	$\{1, x > 1\}$
3	$\{1, x > 1\}$
4	$\{1\}$
5	$\{1, x > 1\}$

Idea: Round Robin Iteration

Instead of accessing the values of the last iteration, always use the current values of unknowns :-)

Example:



	1	2
0	$\emptyset$	
1	$\{1\}$	
2	$\{1, x > 1\}$	
3	$\{1, x > 1\}$	dito
4	$\{1\}$	
5	$\{1, x > 1\}$	