The code for Round Robin Iteration in Java looks as follows:

```
for (i = 1; i ≤ n; i++) x_i = ⊥;
do {
        finished = true;
        for (i = 1; i ≤ n; i++) {
                new = f_i(x_1, ..., x_n);
                if (!(x_i ⊒ new)) {
                        finished = false;
                        x_i = x_i ⊔ new;
                }
        }
} while (!finished);
```

Correctness:

Assume $y_i^{(d)}$ is the i-th component of $F^d \underline{\perp}$.

Assume $x_i^{(d)}$ is the value of x_i after the d-th RR-iteration.

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(3) If RR-iteration terminates after *d* rounds, then $(x_1^{(d)}, \ldots, x_n^{(d)})$ is a solution :-))

Warning:

The efficiency of RR-iteration depends on the ordering of the unknowns !!!

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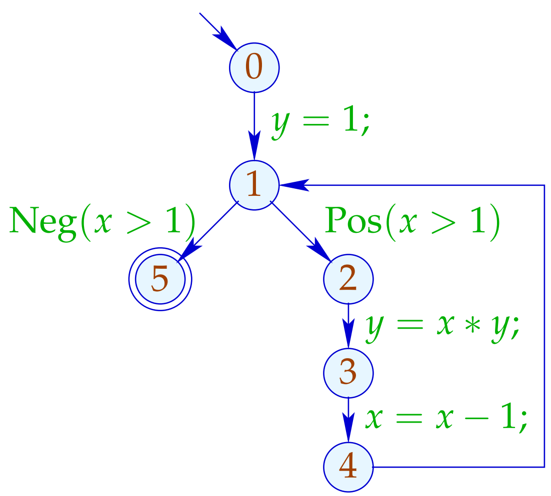
Bad:

e.g., post-order DFS of the CFG, starting at start :-)

Good:



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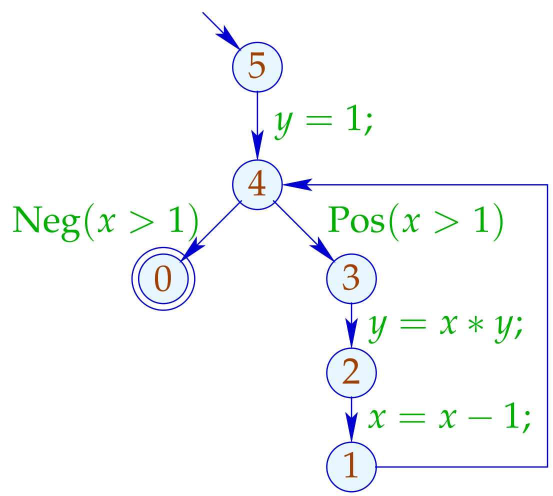
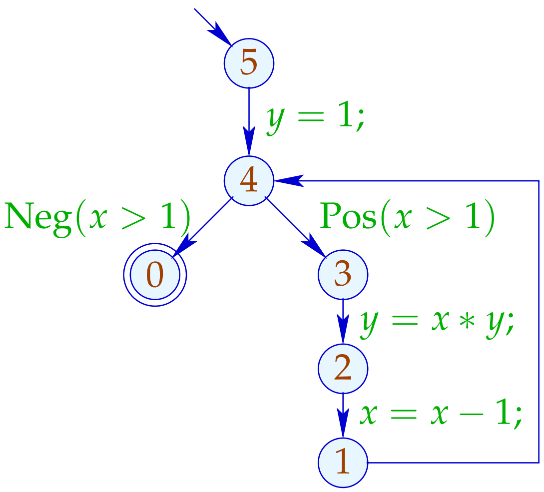


Inefficient Round Robin Iteration:



0
1
2
3
4
5

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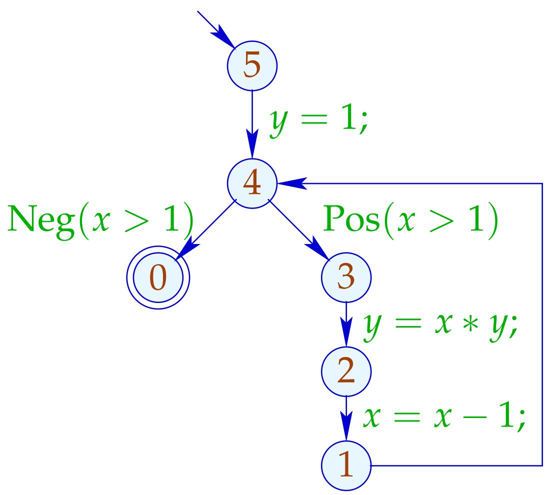

	1
0	*Expr*
1	$\{1\}$
2	$\{1, x - 1, x > 1\}$
3	*Expr*
4	$\{1\}$
5	$\emptyset$

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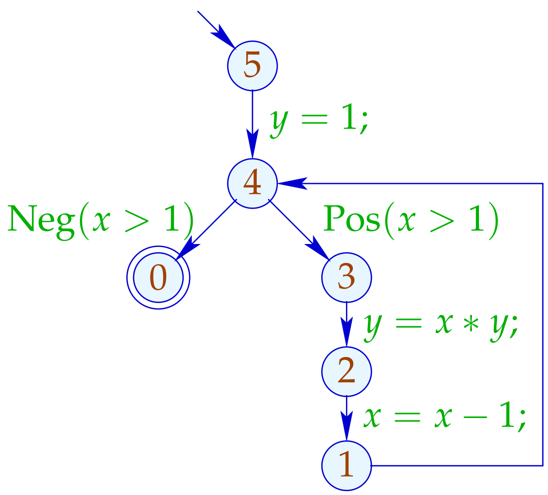

	1	2
0	*Expr*	$\{1, x > 1\}$
1	$\{1\}$	$\{1\}$
2	$\{1, x-1, x > 1\}$	$\{1, x-1, x > 1\}$
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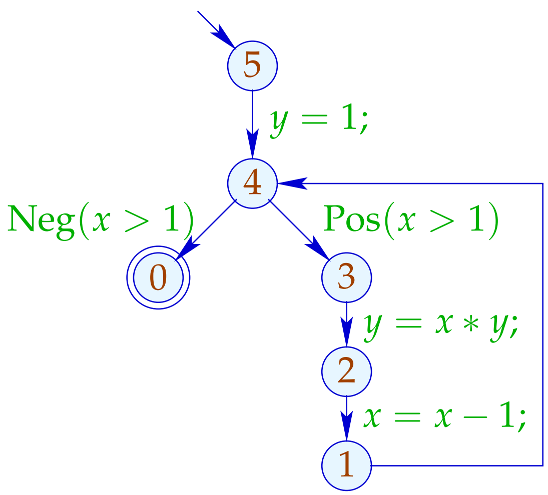

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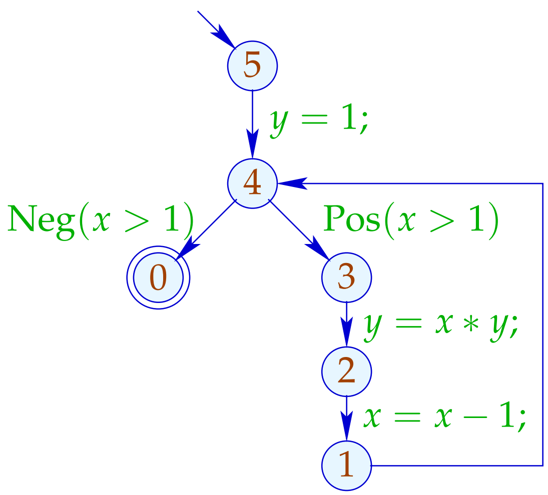

	1	2	3	4
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1	$\{1\}$	$\{1\}$	$\{1\}$	
2	$\{1, x-1, x > 1\}$	$\{1, x-1, x > 1\}$	$\{1, x > 1\}$	dito
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$\Longrightarrow$ significantly less efficient :-)

... end of background on: Complete Lattices

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Final Question:

Why is a (or the least) solution of the constraint system usefull ???

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For a complete lattice $\mathbb{D}$, consider systems:

$$\begin{array}{lll} \mathcal{I}[start] & \sqsupseteq & d_0 \\ \mathcal{I}[v] & \sqsupseteq & [\![k]\!]^\sharp (\mathcal{I}[u]) \qquad k = (u, \_, v) \quad \text{edge} \end{array}$$

where $d_0 \in \mathbb{D}$ and all $[\![k]\!]^\sharp : \mathbb{D} \to \mathbb{D}$ are monotonic ...

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$\Longrightarrow$ Monotonic Analysis Framework

Wanted: MOP (Merge Over all Paths)

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Assume $\mathcal{I}$ is a solution of the constraint system. Then:

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Jeffrey D. Ullman, Stanford

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In particular: $\mathcal{I}[v] \sqsupseteq [\![\pi]\!]^\sharp \, d_0$ for every $\pi : start \rightarrow^* v$

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Then:

$$[\![\pi']\!]^\sharp\, d_0 \quad \sqsubseteq \quad \mathcal{I}[u] \qquad \text{by I.H. for } \pi$$

$$\Longrightarrow \quad [\![\pi]\!]^\sharp\, d_0 \quad = \quad [\![k]\!]^\sharp\, ([\![\pi']\!]^\sharp\, d_0)$$

$$\sqsubseteq \quad [\![k]\!]^\sharp\, (\mathcal{I}[u]) \qquad \text{since } [\![k]\!]^\sharp \text{ monotonic}$$

$$\sqsubseteq \quad \mathcal{I}[v] \qquad \text{since } \mathcal{I} \text{ solution}$$

:-))

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Are solutions of the constraint system just upper bounds ???

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With the notable exception when all functions $[\![k]\!]^\sharp$ are distributive ... :-)

The function $f : \mathbb{D}_1 \to \mathbb{D}_2$ is called

- distributive, if $f\,(\bigsqcup X) = \bigsqcup\{f\,x \mid x \in X\}$ for all $\emptyset \neq X \subseteq \mathbb{D}$;
- strict, if $f\,\bot = \bot$.
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  Distributivity:

$$
\begin{aligned}
f\,(x_1 \cup x_2) &= a \cap (x_1 \cup x_2) \cup b \\
&= a \cap x_1 \cup a \cap x_2 \cup b \\
&= f\,x_1 \cup f\,x_2 \qquad \text{:-)}
\end{aligned}
$$

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  Distributivity:

$$\begin{array}{lllll} f\,((1,4) \sqcup (4,1)) & = & f\,(4,4) & = & 8 \\ & \neq & 5 & = & f\,(1,4) \sqcup f\,(4,1) \end{array}$$:-)

Remark:

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From that follows:

$$\begin{array}{rcl} f\,b & = & f\,(a \sqcup b) \\ & = & f\,a \sqcup f\,b \\ \Longrightarrow \quad f\,a & \sqsubseteq & f\,b \qquad \text{:-)} \end{array}$$

Assumption: all v are reachable from *start* .

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Then:

Theorem

Kildall 1972

If all effects of edges $[\![k]\!]^\sharp$ are distributive, then: $\mathcal{I}^*[v] = \mathcal{I}[v]$
for all v .

Gary A. Kildall (1942-1994).

Has developed the operating system CP/M and GUIs for PCs.

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Proof:

It suffices to prove that $\mathcal{I}^*$ is a solution :-)

For this, we show that $\mathcal{I}^*$ satisfies all constraints :-))

(1) We prove for *start* :

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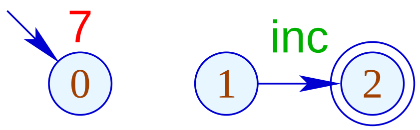
(2) For every $k = (u, \_, v)$ we prove:

$$\begin{aligned}\mathcal{I}^*[v] &= \bigsqcup\{[\![\pi]\!]^\sharp\, d_0 \mid \pi : start \to^* v\} \\ &\sqsupseteq \bigsqcup\{[\![\pi' k]\!]^\sharp\, d_0 \mid \pi' : start \to^* u\} \\ &= \bigsqcup\{[\![k]\!]^\sharp\, ([\![\pi']\!]^\sharp\, d_0) \mid \pi' : start \to^* u\} \\ &= [\![k]\!]^\sharp\, (\bigsqcup\{[\![\pi']\!]^\sharp\, d_0 \mid \pi' : start \to^* u\}) \\ &= [\![k]\!]^\sharp\, (\mathcal{I}^*[u])\end{aligned}$$

since $\{\pi' \mid \pi' : start \to^* u\}$ is non-empty :-)

Warning:

- Reachability of all program points cannot be abandoned!
  Consider:



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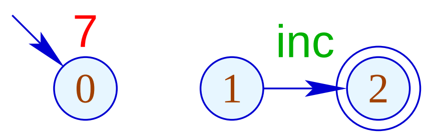

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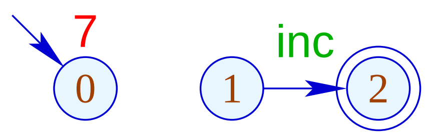

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$$\begin{array}{lllll} \mathcal{I}[2] & = & \mathsf{inc}\,0 & = & 1 \\ \mathcal{I}^*[2] & = & \bigsqcup \emptyset & = & 0 \end{array}$$

- Unreachable program points can always be thrown away :-)

Summary and Application:

→ The effects of edges of the analysis of availability of expressions are distributive:

$$\begin{aligned}(a \cup (x_1 \cap x_2))\backslash b &= ((a \cup x_1) \cap (a \cup x_2))\backslash b \\ &= ((a \cup x_1)\backslash b) \cap ((a \cup x_2)\backslash b)\end{aligned}$$

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→ If all effects of edges are distributive, then the MOP can be computed by means of the constraint system and RR-iteration. :-)

→ If not all effects of edges are distributive, then RR-iteration for the constraint system at least returns a safe upper bound to the MOP :-)

1.2 Removing Assignments to Dead Variables

Example:

$$\begin{array}{ll} 1: & x = y + 2; \\ 2: & y = 5; \\ 3: & x = y + 3; \end{array}$$

The value of x at program points $1, 2$ is over-written before it can be used.

Therefore, we call the variable x dead at these program points :-)

Note:

→ Assignments to dead variables can be removed ;-)

→ Such inefficiencies may originate from other transformations.

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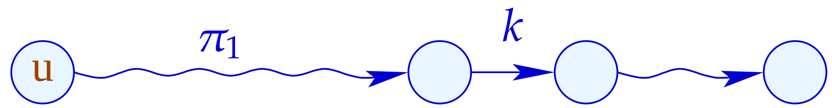
Formal Definition:

The variable x is called live at u along the path π starting at u relative to a set X of variables either:

if $x \in X$ and π does not contain a definition of x; or:

if π can be decomposed into: $\pi = \pi_1 \, k \, \pi_2$ such that:

- k is a use of x ; and
- π_1 does not contain a definition of x.

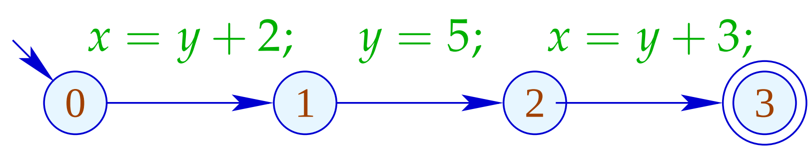


Thereby, the set of all defined or used variables at an edge $k = (\_, lab, \_)$ is defined by:

lab	used	defined
$;$	$\emptyset$	$\emptyset$
$\text{Pos}\,(e)$	$Vars\,(e)$	$\emptyset$
$\text{Neg}\,(e)$	$Vars\,(e)$	$\emptyset$
$x = e;$	$Vars\,(e)$	$\{x\}$
$x = M[e];$	$Vars\,(e)$	$\{x\}$
$M[e_1] = e_2;$	$Vars\,(e_1) \cup Vars\,(e_2)$	$\emptyset$

A variable x which is not live at u along π (relative to X) is called dead at u along π (relative to X).

Example:



where $X = \emptyset$. Then we observe:

	live	dead
0	$\{y\}$	$\{x\}$
1	$\emptyset$	$\{x, y\}$
2	$\{y\}$	$\{x\}$
3	$\emptyset$	$\{x, y\}$

The variable x is live at u (relative to X) if x is live at u along some path to the exit (relative to X). Otherwise, x is called dead at u (relative to X).

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Question:

How can the sets of all dead/live variables be computed for every u ???

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Idea:

For every edge $k = (u, \_, v)$, define a function $[\![k]\!]^\sharp$ which transforms the set of variables which are live at v into the set of variables which are live at u ...