We conclude:

→ Solving the constraint system returns the MOP solution :-)

→ Let $\mathcal{V}$ denote this solution.

If $x \in \mathcal{V}[u]\,e$, then x at u contains the value of e — which we have stored in T_e

$\Longrightarrow$

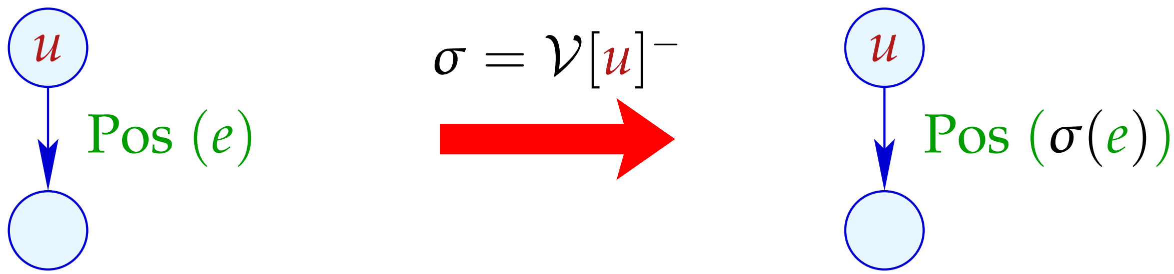
the access to x can be replaced by the access to T_e :-)

For $V \in \mathbb{V}$, let V^- denote the variable substitution with:

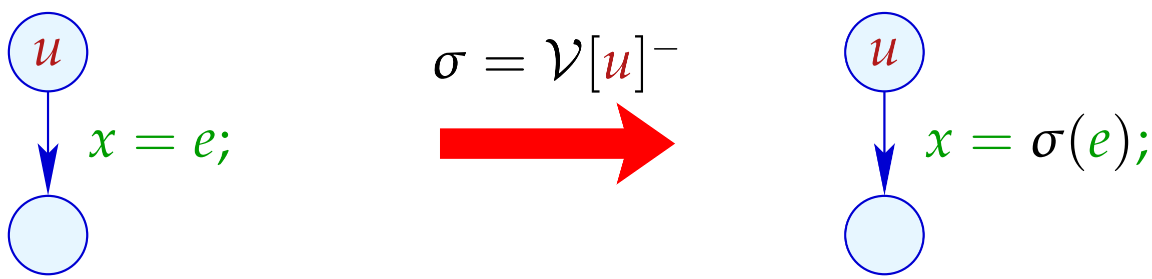
$$V^-\,x = \begin{cases} T_e & \text{if } x \in V\,e \\ x & \text{otherwise} \end{cases}$$

if $V\,e \cap V\,e' = \emptyset$ for $e \neq e'$. Otherwise: $V^-\,x = x$:-)

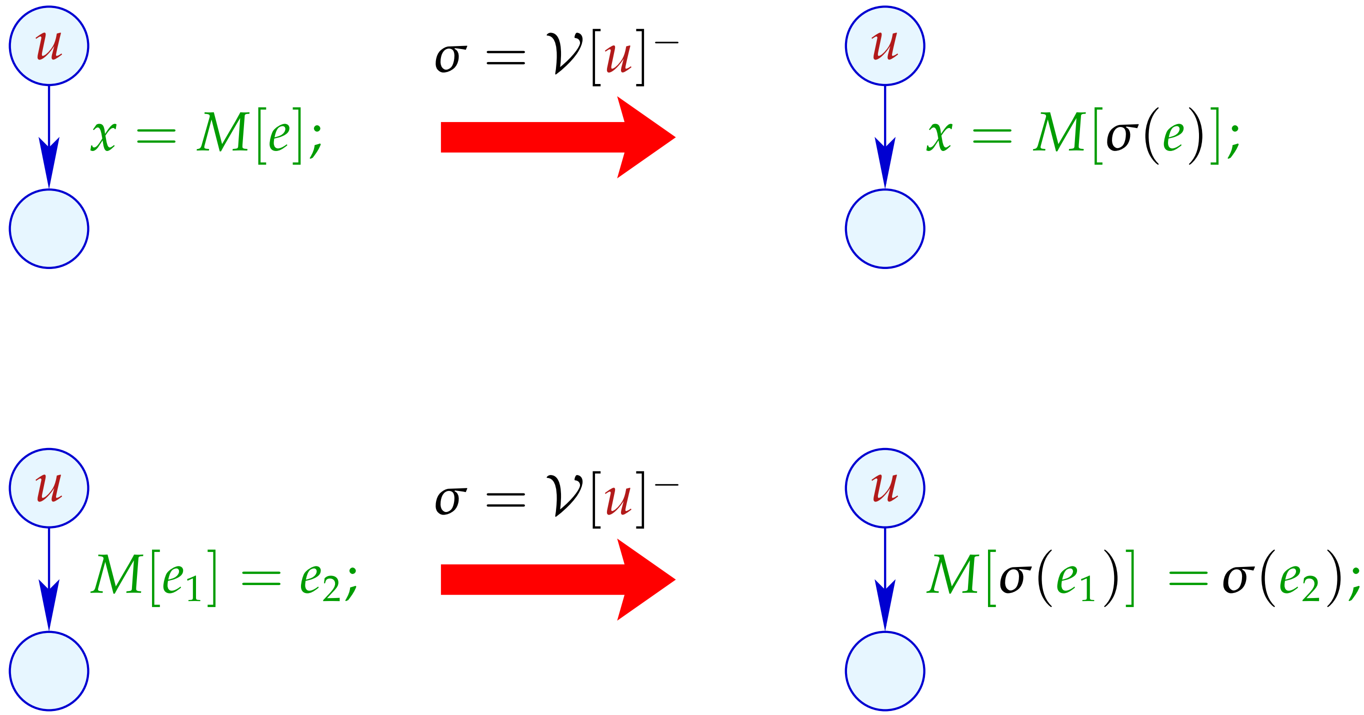
Transformation 3:



... analogously for edges with Neg (e)



Transformation 3 (cont.):



Procedure as a whole:

(1) Availability of expressions: T1

- \+ removes arithmetic operations
- − inserts superfluous moves

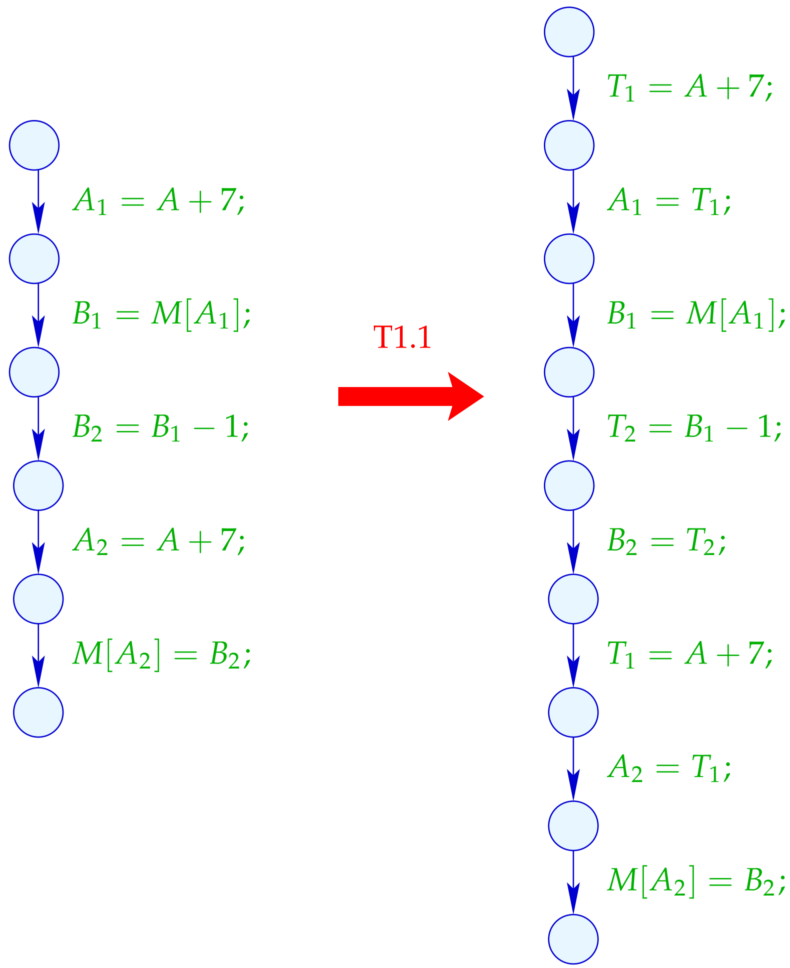
(2) Values of variables: T3

- \+ creates dead variables

(3) (true) liveness of variables: T2

- \+ removes assignments to dead variables

Example: `a[7]--;`


$A_1 = A + 7;$
$B_1 = M[A_1];$
$B_2 = B_1 - 1;$
$A_2 = A + 7;$
$M[A_2] = B_2;$
T1.1
$T_1 = A + 7;$
$A_1 = T_1;$
$B_1 = M[A_1];$
$T_2 = B_1 - 1;$
$B_2 = T_2;$
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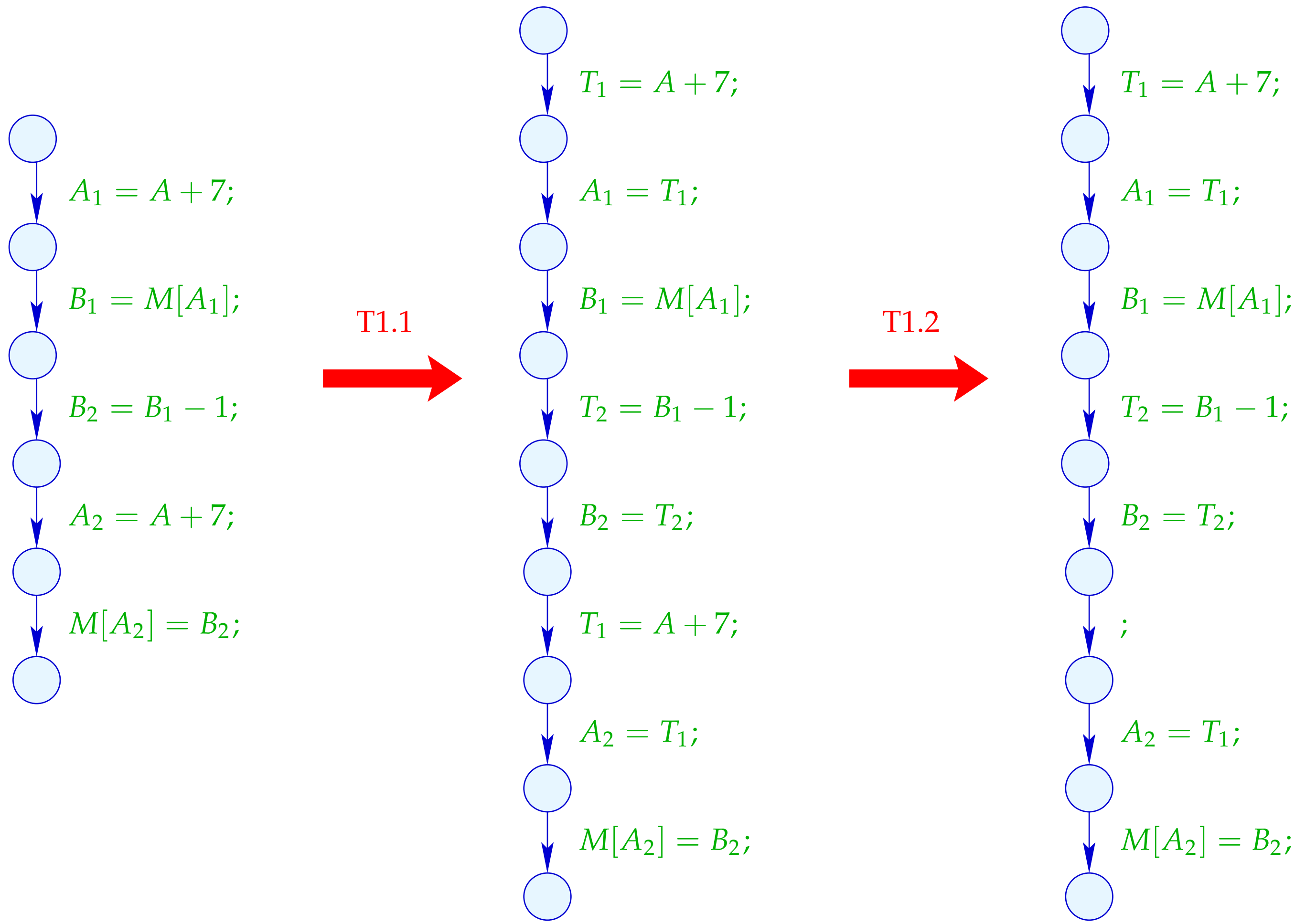
$A_1 = A + 7;$
$B_1 = M[A_1];$
$B_2 = B_1 - 1;$
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$B_2 = T_2;$
$T_1 = A + 7;$
$A_2 = T_1;$
$M[A_2] = B_2;$
T1.2
$T_1 = A + 7;$
$A_1 = T_1;$
$B_1 = M[A_1];$
$T_2 = B_1 - 1;$
$B_2 = T_2;$
$;$
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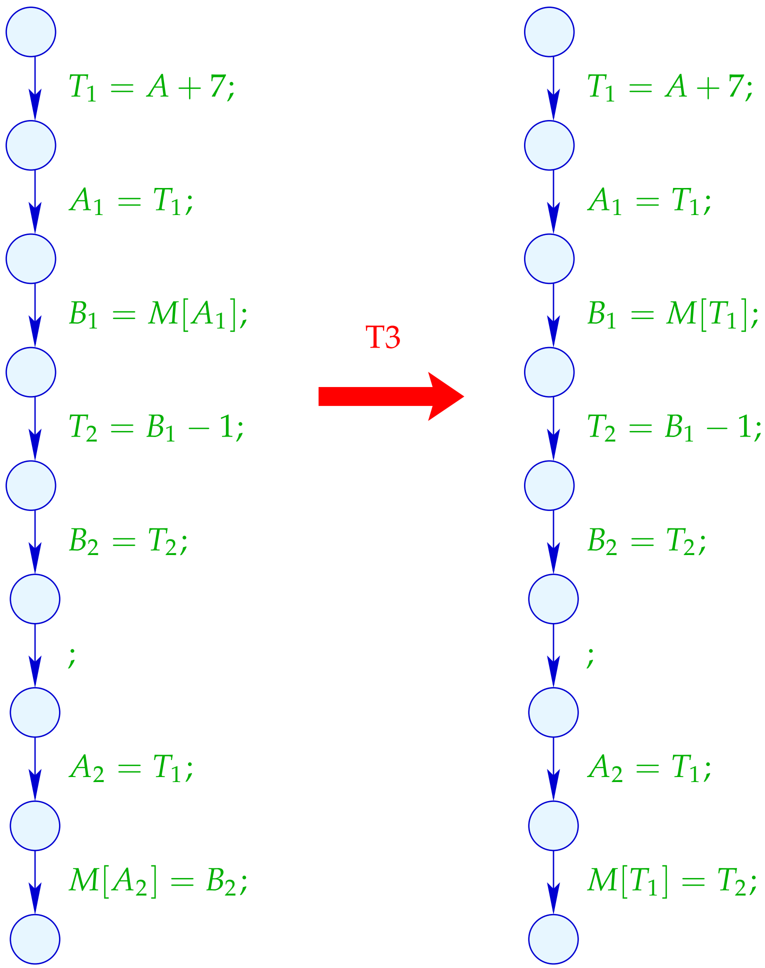
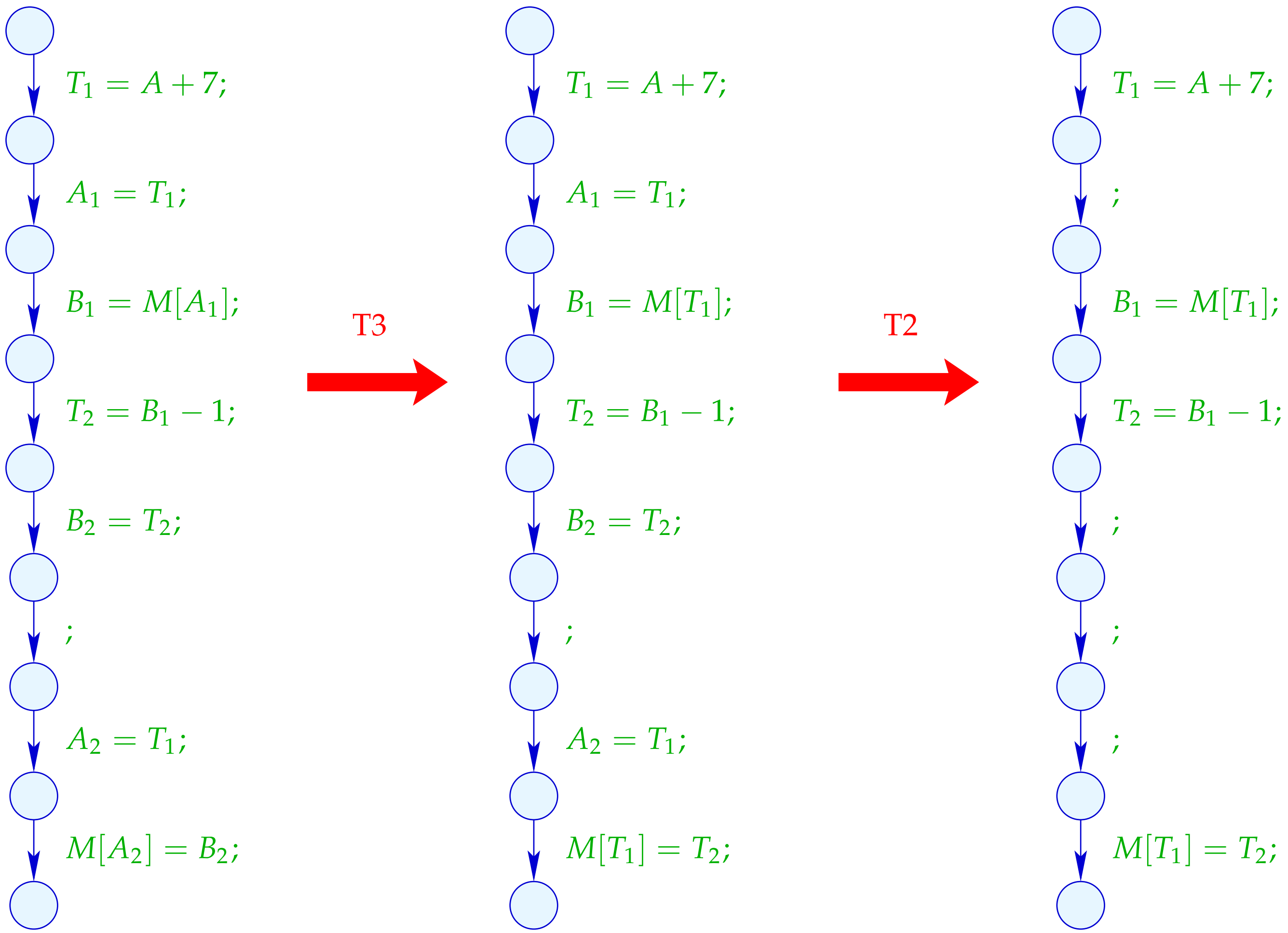
Example (cont.): `a[7]--;`



Example (cont.): `a[7]--;`



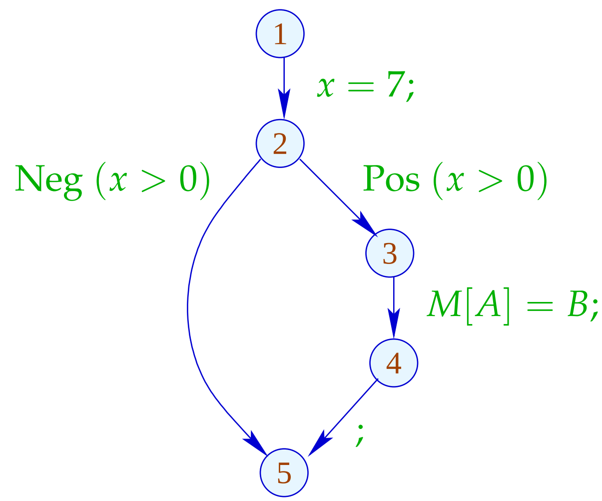
1.4 Constant Propagation

Idea:

Execute as much of the code at compile-time as possible!

Example:

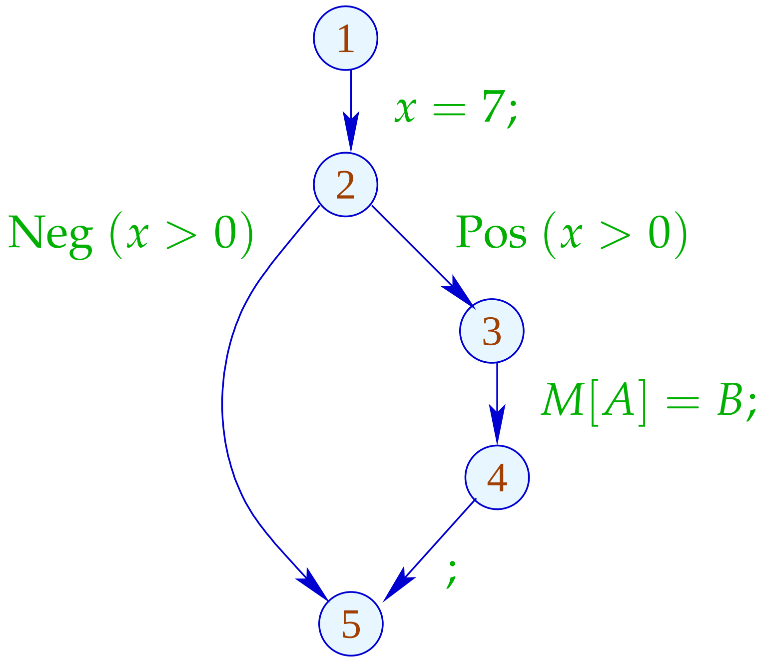
$$x = 7;$$
$$\text{if } (x > 0)$$
$$\quad M[A] = B;$$



Obviously, x has always the value 7 :-)

Thus, the memory access is always executed :-))

Goal:



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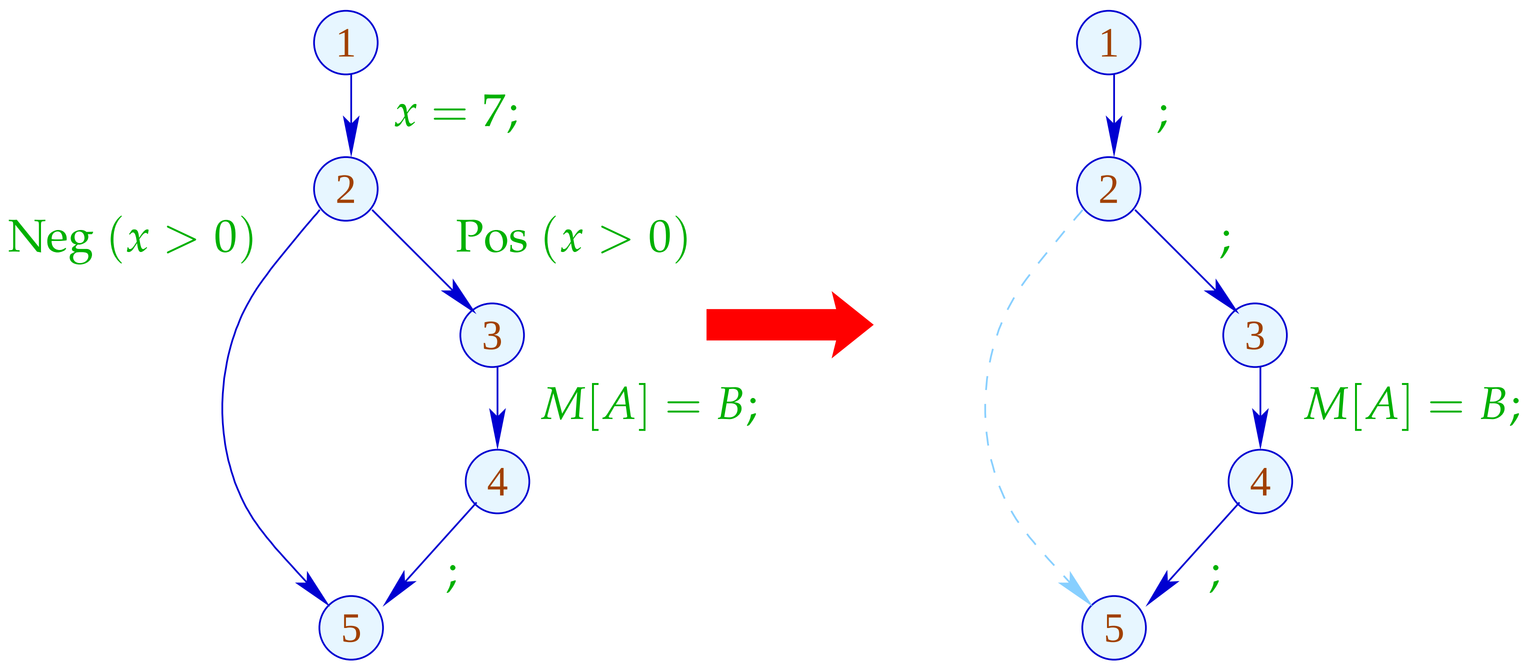

Generalization: Partial Evaluation

Neil D. Jones, DIKU, Kopenhagen

Idea:

Design an analysis which for every u,

- determines the values which variables definitely have;
- tells whether u can be reached at all :-)

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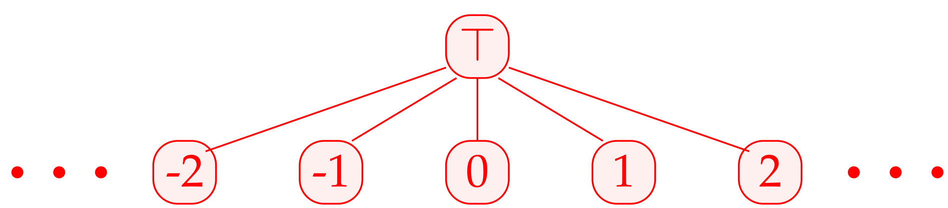
Design an analysis which for every u,

- determines the values which variables definitely have;
- tells whether u can be reached at all :-)

The complete lattice is constructed in two steps.

(1) The potential values of variables:

$$\mathbb{Z}^\top = \mathbb{Z} \cup \{\top\} \qquad \text{with} \quad x \sqsubseteq y \quad \text{iff } y = \top \text{ or } x = y$$



Warning: $\mathbb{Z}^\top$ is not a complete lattice in itself :-(

(2) $\mathbb{D} = (\mathit{Vars} \to \mathbb{Z}^\top)_\bot = (\mathit{Vars} \to \mathbb{Z}^\top) \cup \{\bot\}$

// $\bot$ denotes: "not reachable" :-))

with $D_1 \sqsubseteq D_2$ iff $\bot = D_1$ or

$D_1\, x \sqsubseteq D_2\, x \quad (x \in \mathit{Vars})$

Remark: $\mathbb{D}$ is a complete lattice :-)

Warning: $\mathbb{Z}^\top$ is not a complete lattice in itself :-(

(2) $\mathbb{D} = (\mathit{Vars} \to \mathbb{Z}^\top)_\perp = (\mathit{Vars} \to \mathbb{Z}^\top) \cup \{\perp\}$

// $\perp$ denotes: "not reachable" :-))

with $D_1 \sqsubseteq D_2$ iff $\perp = D_1$ or

$D_1\,x \sqsubseteq D_2\,x \quad (x \in \mathit{Vars})$

Remark: $\mathbb{D}$ is a complete lattice :-)

Consider $X \subseteq \mathbb{D}$. W.l.o.g., $\perp \not\in X$.

Then $X \subseteq \mathit{Vars} \to \mathbb{Z}^\top$.

If $X = \emptyset$, then $\bigsqcup X = \perp \in \mathbb{D}$:-)

If $X \neq \emptyset$, then $\bigsqcup X = D$ with

$$\begin{array}{rcl} D\,x & = & \bigsqcup\{f\,x \mid f \in X\} \\ & = & \begin{cases} z & \text{if} \quad f\,x = z \quad (f \in X) \\ \top & \text{otherwise} \end{cases} \end{array}$$

:-))

If $X \neq \emptyset$, then $\bigsqcup X = D$ with

$$D\,x = \bigsqcup\{f\,x \mid f \in X\} = \begin{cases} z & \text{if} \quad f\,x = z \quad (f \in X) \\ \top & \text{otherwise} \end{cases}$$

:-))

For every edge $k = (\_, lab, \_)$, construct an effect function $[\![k]\!]^\sharp = [\![lab]\!]^\sharp : \mathbb{D} \to \mathbb{D}$ which simulates the concrete computation.

Obviously, $[\![lab]\!]^\sharp \perp = \perp$ for all lab :-)

Now let $\perp \neq D \in Vars \to \mathbb{Z}^\top$.

Idea:

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We must replace the concrete operators $\Box$ by abstract operators $\Box^\sharp$ which can handle $\top$:

$$a \Box^\sharp b = \begin{cases} \top & \text{if} \quad a = \top \text{ or } b = \top \\ a \Box b & \text{otherwise} \end{cases}$$

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$$a \,\Box^\sharp\, b = \begin{cases} \top & \text{if} \quad a = \top \text{ or } b = \top \\ a \,\Box\, b & \text{otherwise} \end{cases}$$

- The abstract operators allow to define an abstract evaluation of expressions:

$$[\![e]\!]^\sharp \;:\; (\mathit{Vars} \to \mathbb{Z}^\top) \to \mathbb{Z}^\top$$

Abstract evaluation of expressions is like the concrete evaluation — but with abstract values and operators. Here:

$$
\begin{array}{lll}
[\![c]\!]^\sharp\, D & = & c \\
[\![e_1 \,\Box\, e_2]\!]^\sharp\, D & = & [\![e_1]\!]^\sharp\, D \,\Box^\sharp\, [\![e_2]\!]^\sharp\, D
\end{array}
$$

... analogously for unary operators :-)

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... analogously for unary operators :-)

Example: $D = \{x \mapsto 2, y \mapsto \top\}$

$$\begin{array}{lll} [\![x+7]\!]^\sharp\, D & = & [\![x]\!]^\sharp\, D +^\sharp [\![7]\!]^\sharp\, D \\ & = & 2 +^\sharp 7 \\ & = & 9 \\ [\![x-y]\!]^\sharp\, D & = & 2 -^\sharp \top \\ & = & \top \end{array}$$

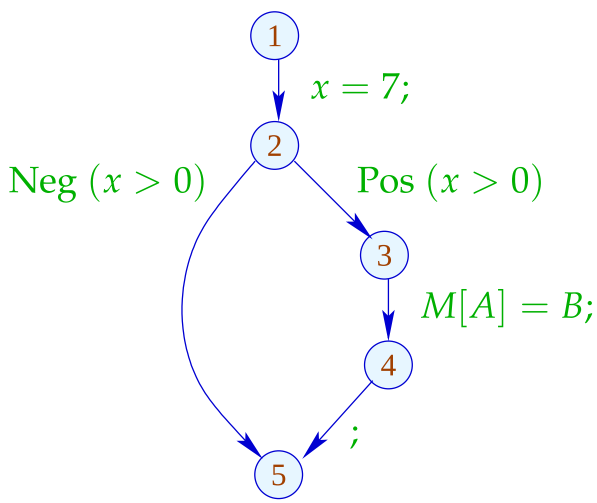
Thus, we obtain the following effects of edges $[\![lab]\!]^\sharp$:

$$
\begin{array}{lll}
[\![;]\!]^\sharp\, D & = & D \\
[\![\text{Pos}\,(e)]\!]^\sharp\, D & = & \left\{ \begin{array}{ll} \bot & \text{if} \quad 0 = [\![e]\!]^\sharp\, D \\ D & \text{otherwise} \end{array} \right. \\
[\![\text{Neg}\,(e)]\!]^\sharp\, D & = & \left\{ \begin{array}{ll} D & \text{if} \quad 0 \sqsubseteq [\![e]\!]^\sharp\, D \\ \bot & \text{otherwise} \end{array} \right. \\
[\![x = e;]\!]^\sharp\, D & = & D \oplus \{x \mapsto [\![e]\!]^\sharp\, D\} \\
[\![x = M[e];]\!]^\sharp\, D & = & D \oplus \{x \mapsto \top\} \\
[\![M[e_1] = e_2;]\!]^\sharp\, D & = & D
\end{array}
$$

... whenever $D \neq \bot$:-)

At *start*, we have $D_\top = \{x \mapsto \top \mid x \in Vars\}$.

Example:



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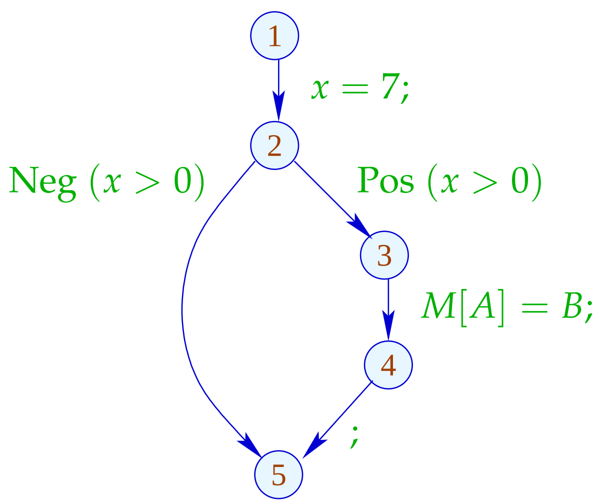

1	$\{x \mapsto \top\}$
2	$\{x \mapsto 7\}$
3	$\{x \mapsto 7\}$
4	$\{x \mapsto 7\}$
5	$\bot \sqcup \{x \mapsto 7\} = \{x \mapsto 7\}$

The abstract effects of edges $[\![k]\!]^\sharp$ are again composed to the effects of paths $\pi = k_1 \dots k_r$ by:

$$[\![\pi]\!]^\sharp = [\![k_r]\!]^\sharp \circ \dots \circ [\![k_1]\!]^\sharp \quad : \mathbb{D} \to \mathbb{D}$$

Idea for Correctness: Abstract Interpretation

Cousot, Cousot 1977

Patrick Cousot, ENS, Paris

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Idea for Correctness: Abstract Interpretation

Cousot, Cousot 1977

Establish a description relation Δ between theconcrete values and their descriptions with:

$$x \,\Delta\, a_1 \quad \wedge \quad a_1 \sqsubseteq a_2 \quad \Longrightarrow \quad x \,\Delta\, a_2$$

Concretization: $\gamma\, a = \{x \mid x \,\Delta\, a\}$

// returns the set of described values :-)

(1) Values: $\Delta \subseteq \mathbb{Z} \times \mathbb{Z}^{\top}$

$$z \, \Delta \, a \quad \text{iff} \quad z = a \vee a = \top$$

Concretization:

$$\gamma\, a = \begin{cases} \{a\} & \text{if} \quad a \sqsubset \top \\ \mathbb{Z} & \text{if} \quad a = \top \end{cases}$$

(1) Values: $\Delta \subseteq \mathbb{Z} \times \mathbb{Z}^\top$

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(2) Variable Assignments: $\Delta \subseteq (\mathit{Vars} \to \mathbb{Z}) \times (\mathit{Vars} \to \mathbb{Z}^\top)_\bot$

$$\rho \,\Delta\, D \quad \text{iff} \quad D \neq \bot \wedge \rho\, x \sqsubseteq D\, x \quad (x \in \mathit{Vars})$$

Concretization:

$$\gamma\, D = \begin{cases} \emptyset & \text{if} \quad D = \bot \\ \{\rho \mid \forall\, x : (\rho\, x) \,\Delta\, (D\, x)\} & \text{otherwise} \end{cases}$$

Example: $\{x \mapsto 1, y \mapsto -7\} \ \Delta \ \{x \mapsto \top, y \mapsto -7\}$

(3) States:

$$\Delta \subseteq ((Vars \to \mathbb{Z}) \times (\mathbb{N} \to \mathbb{Z})) \times (Vars \to \mathbb{Z}^\top)_\bot$$

$$(\rho, \mu) \ \Delta \ D \qquad \text{gdw.} \qquad \rho \ \Delta \ D$$

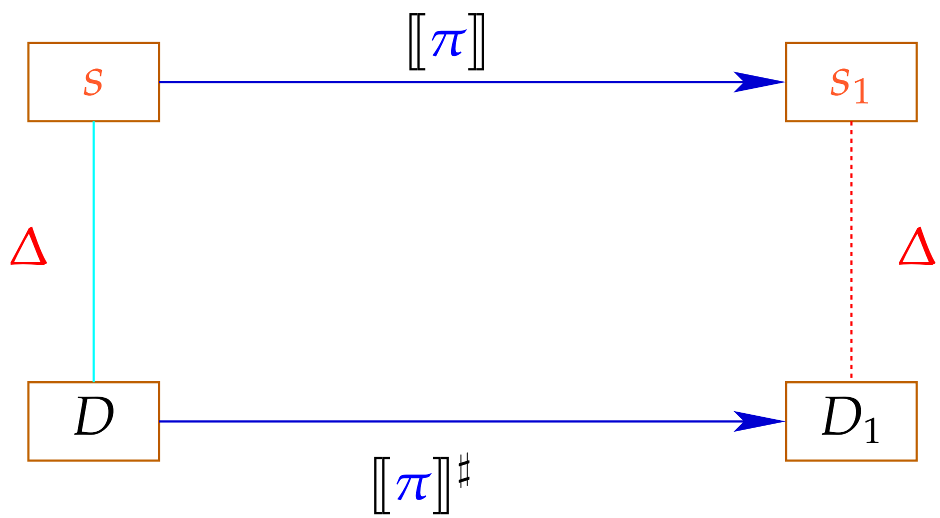
Concretization:

$$\gamma\, D = \begin{cases} \emptyset & \text{if} \quad D = \bot \\ \{(\rho, \mu) \mid \forall\, x : \ (\rho\, x) \ \Delta \ (D\, x)\} & \text{otherwise} \end{cases}$$

We show:

(∗) If $s \,\Delta\, D$ and $[\![\pi]\!]\, s$ is defined, then:

$$([\![\pi]\!]\, s) \;\Delta\; ([\![\pi]\!]^\sharp\, D)$$



The abstract semantics simulates the die concrete semantics

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In particular:

$$[\![\pi]\!]\, s \in \gamma\left([\![\pi]\!]^\sharp\, D\right)$$

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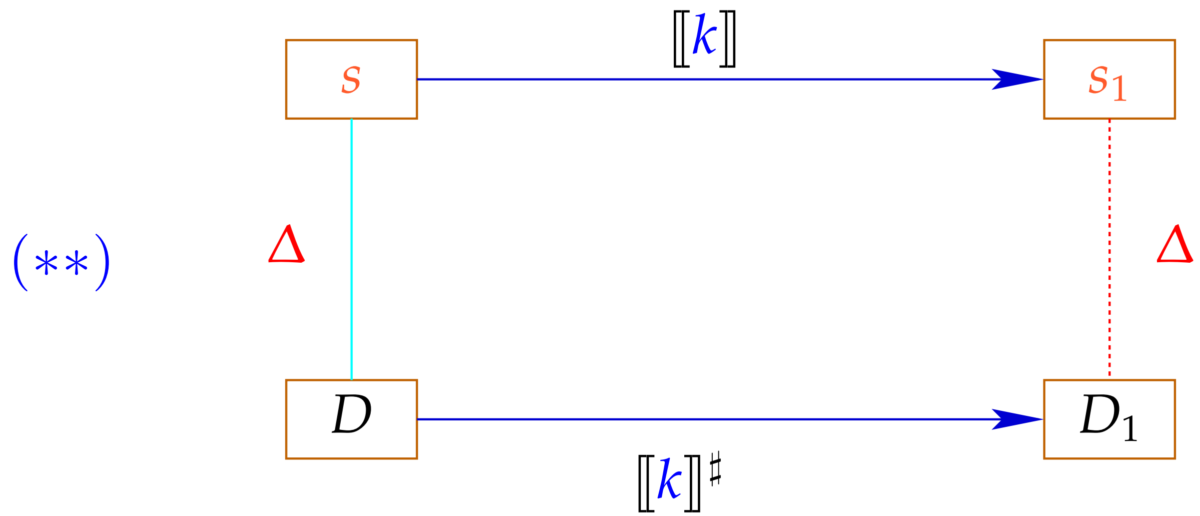
In particular:

$$[\![\pi]\!]\, s \in \gamma\, ([\![\pi]\!]^\sharp\, D)$$

In practice, this means, e.g., that $D\, x = -7$ implies:

$$\begin{array}{lllll} & \rho'\, x & = & -7 & \text{for all} \quad \rho' \in \gamma\, D \\ \Longrightarrow & \rho_1\, x & = & -7 & \text{for} \quad (\rho_1, \_) = [\![\pi]\!]\, s \end{array}$$

To prove $(*)$, we show for every edge k :



Then $(*)$ follows by induction :-)

To prove $(**)$, we show for every expression e :

$(***)$ $\quad ([\![e]\!]\,\rho) \;\;\Delta\;\; ([\![e]\!]^\sharp\, D) \quad$ whenever $\quad \rho \;\Delta\; D$

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$$(x \Box y) \;\Delta\; (x^\sharp \Box^\sharp y^\sharp) \qquad \text{whenever} \quad x \,\Delta\, x^\sharp \wedge y \,\Delta\, y^\sharp$$

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This precisely was how we have defined the operators $\Box^\sharp$:-)

Now, $(**)$ is proved by case distinction on the edge labels *lab* .

Let $s = (\rho, \mu) \;\Delta\; D$. In particular, $\bot \neq D \;:\; Vars \to \mathbb{Z}^\top$

Case $\boxed{x = e;}$:

$$\rho_1 = \rho \oplus \{x \mapsto [\![e]\!]\,\rho\} \qquad \mu_1 = \mu$$
$$D_1 = D \oplus \{x \mapsto [\![e]\!]^\sharp\, D\}$$

$$\Longrightarrow \qquad (\rho_1, \mu_1) \;\Delta\; D_1$$

Case $\boxed{x = M[e];}$:

$$\rho_1 = \rho \oplus \{x \mapsto \mu\,(\llbracket e \rrbracket^\sharp \rho)\} \qquad \mu_1 = \mu$$

$$D_1 = D \oplus \{x \mapsto \top\}$$

$$\Longrightarrow \qquad (\rho_1, \mu_1)\ \Delta\ D_1$$

Case $\boxed{M[e_1] = e_2;}$:

$$\rho_1 = \rho \qquad \mu_1 = \mu \oplus \{\llbracket e_1 \rrbracket^\sharp \rho \mapsto \llbracket e_2 \rrbracket^\sharp \rho\}$$

$$D_1 = D$$

$$\Longrightarrow \qquad (\rho_1, \mu_1)\ \Delta\ D_1$$

Case $\boxed{\mathrm{Neg}(e)}$: $(\rho_1, \mu_1) = s$ where:

$$
\begin{array}{rrcl}
 & 0 & = & [\![e]\!]\,\rho \\
 & & \Delta & [\![e]\!]^\sharp\,D \\
\Longrightarrow & 0 & \sqsubseteq & [\![e]\!]^\sharp\,D \\
\Longrightarrow & \bot & \neq & D_1 \;=\; D \\
\Longrightarrow & \multicolumn{3}{l}{(\rho_1, \mu_1)\;\Delta\;D_1}
\end{array}
$$

Case $\boxed{\text{Pos}(e)}$: $(\rho_1, \mu_1) = s$ where:

$$
\begin{array}{rrcl}
 & 0 & \neq & [\![e]\!]\,\rho \\
 & & \Delta & [\![e]\!]^{\sharp}\,D \\
\Longrightarrow & 0 & \neq & [\![e]\!]^{\sharp}\,D \\
\Longrightarrow & \bot & \neq & D_1 \;=\; D \\
\Longrightarrow & \multicolumn{3}{l}{(\rho_1, \mu_1)\ \Delta\ D_1}
\end{array}
$$

:-)

We conclude: The assertion $(*)$ is true :-))

The MOP-Solution:

$$\mathcal{D}^*[v] \;=\; \bigsqcup \{ [\![\pi]\!]^\sharp\, D_\top \mid \pi : start \rightarrow^* v \}$$

where $\qquad D_\top\, x = \top \qquad (x \in Vars)$.

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By $(*)$, we have for all initial states s and all program executions π which reach v :

$$([\![\pi]\!]\, s) \;\;\Delta\;\; (\mathcal{D}^*[v])$$

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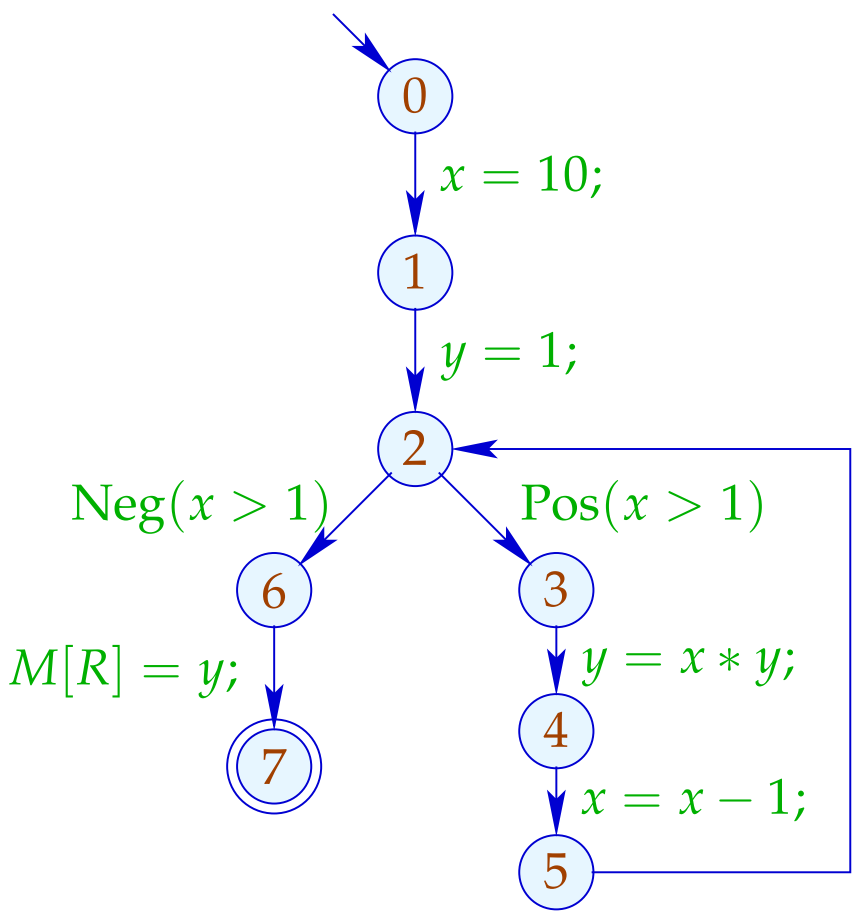
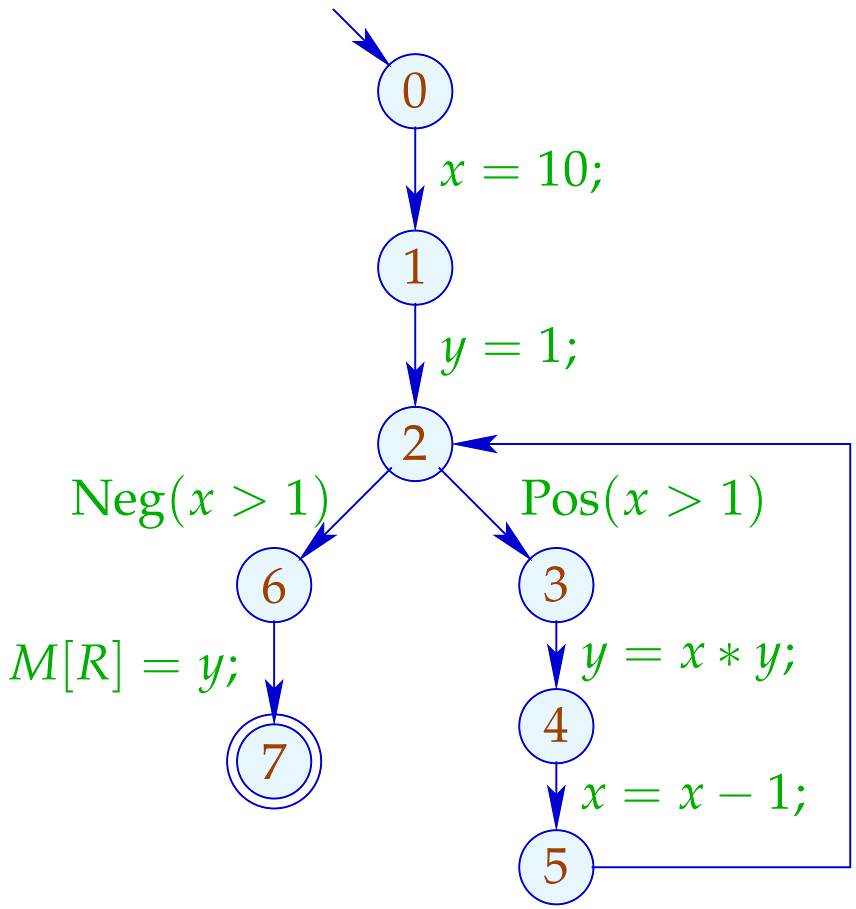
In order to approximate the MOP, we use our constraint system :-))

Example:

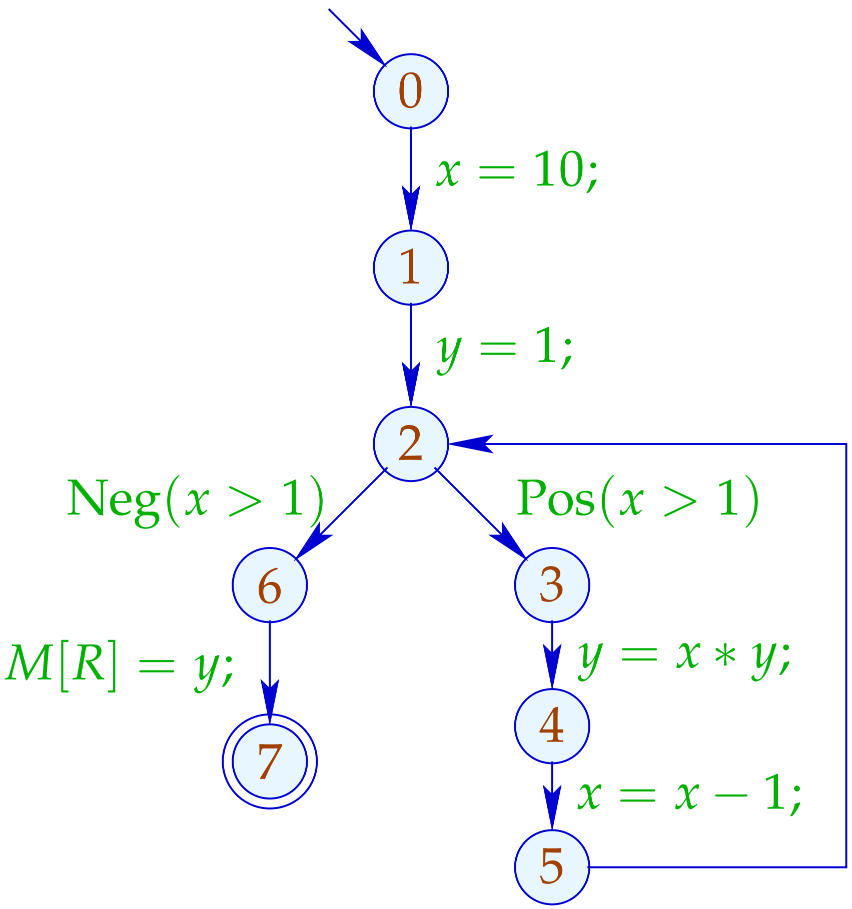


Example:



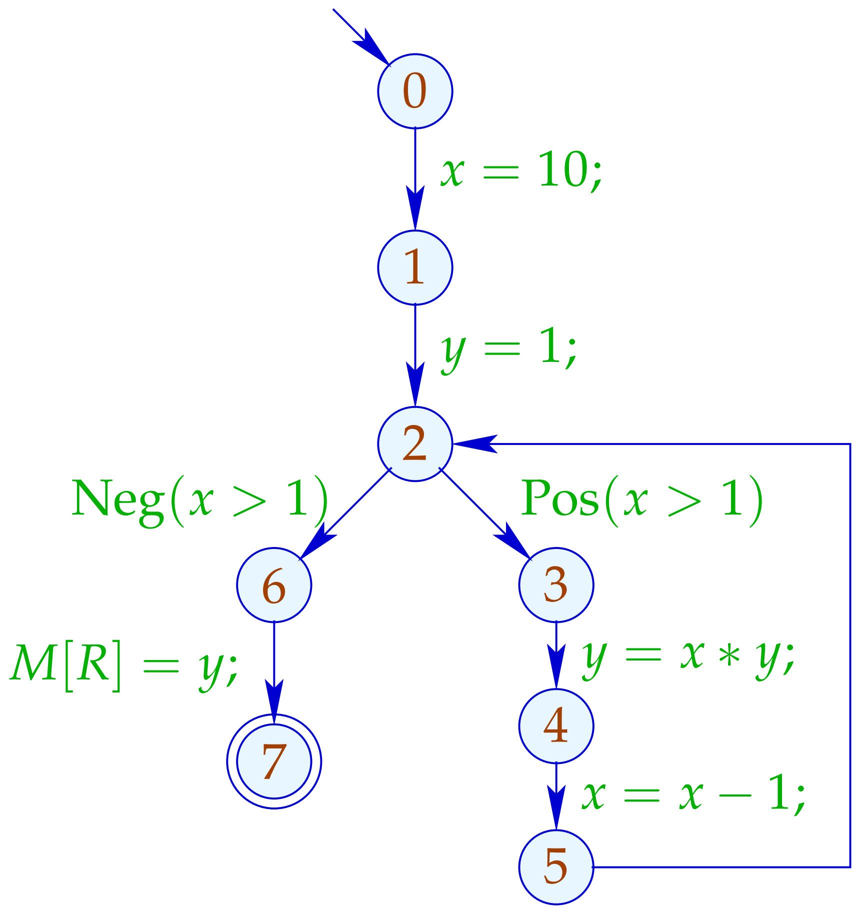
	1	
	x	y
0	$\top$	$\top$
1	10	$\top$
2	10	1
3	10	1
4	10	10
5	9	10
6	$\bot$	
7	$\bot$	

Example:



	1		2	
	x	y	x	y
0	$\top$	$\top$	$\top$	$\top$
1	10	$\top$	10	$\top$
2	10	1	$\top$	$\top$
3	10	1	$\top$	$\top$
4	10	10	$\top$	$\top$
5	9	10	$\top$	$\top$
6	$\bot$		$\top$	$\top$
7	$\bot$		$\top$	$\top$

Example:



	1		2		3	
	x	y	x	y	x	y
0	$\top$	$\top$	$\top$	$\top$		
1	10	$\top$	10	$\top$		
2	10	1	$\top$	$\top$		
3	10	1	$\top$	$\top$		
4	10	10	$\top$	$\top$	dito	
5	9	10	$\top$	$\top$		
6	$\bot$		$\top$	$\top$		
7	$\bot$		$\top$	$\top$		

Conclusion:

Although we compute with concrete values, we fail to compute everything :-(

The fixpoint iteration, at least, is guaranteed to terminate:

For n program points and m variables, we maximally need: $n \cdot (m+1)$ rounds :-)

Warning:

The effects of edge are not distributive !!!

Counter Example: $f = [\![x = x + y;]\!]^\sharp$

$$\begin{array}{lrcl} \text{Let} & D_1 & = & \{x \mapsto 2, y \mapsto 3\} \\ & D_2 & = & \{x \mapsto 3, y \mapsto 2\} \\ \text{Dann} & f\,D_1 \sqcup f\,D_2 & = & \{x \mapsto 5, y \mapsto 3\} \sqcup \{x \mapsto 5, y \mapsto 2\} \\ & & = & \{x \mapsto 5, y \mapsto \top\} \\ & & \neq & \{x \mapsto \top, y \mapsto \top\} \\ & & = & f\,\{x \mapsto \top, y \mapsto \top\} \\ & & = & f\,(D_1 \sqcup D_2) \end{array}$$

:-((

We conclude:

The least solution $\mathcal{D}$ of the constraint system in general yields only an upper approximation of the MOP, i.e.,

$$\mathcal{D}^*[v] \quad \sqsubseteq \quad \mathcal{D}[v]$$

We conclude:

The least solution $\mathcal{D}$ of the constraint system in general yields only an upper approximation of the MOP, i.e.,

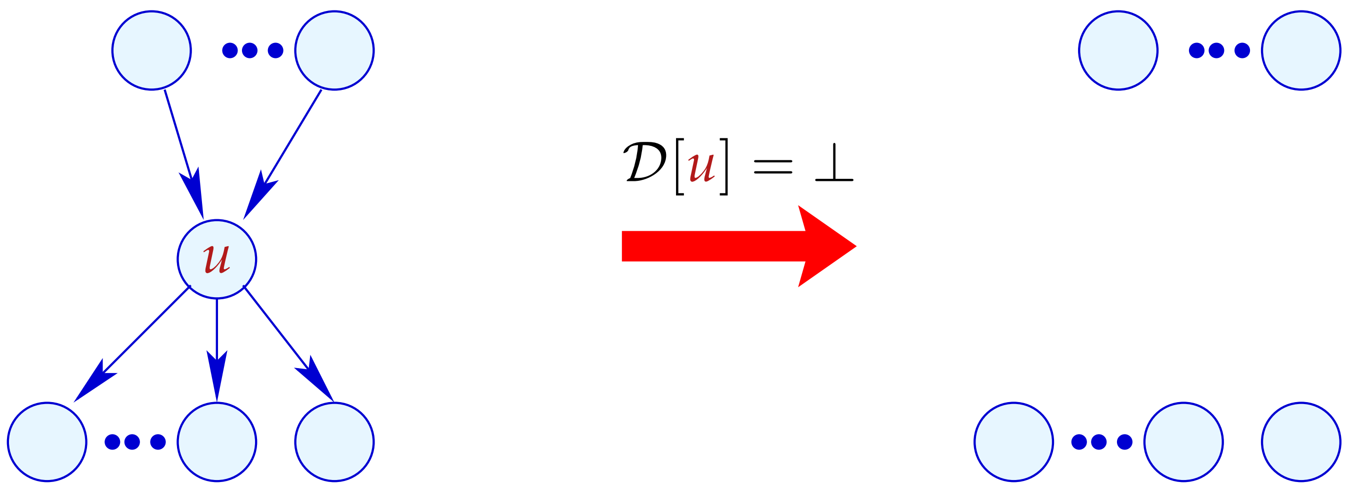
$$\mathcal{D}^*[v] \quad \sqsubseteq \quad \mathcal{D}[v]$$

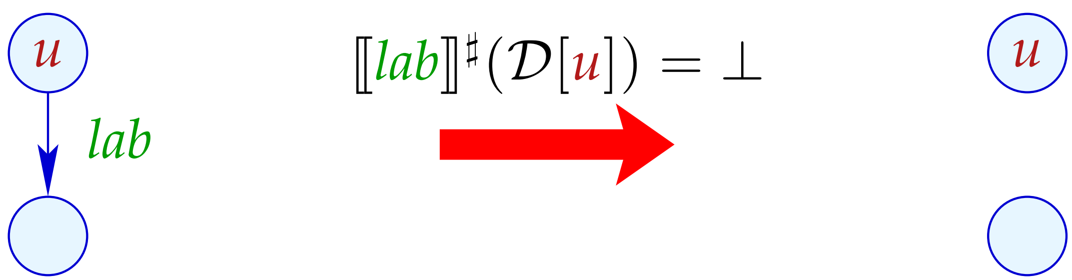
As an upper approximation, $\mathcal{D}[v]$ nonetheless describes the result of every program execution π which reaches v :

$$([\![\pi]\!]\,(\rho, \mu)) \;\; \Delta \;\; (\mathcal{D}[v])$$

whenever $[\![\pi]\!]\,(\rho, \mu)$ is defined ;-))

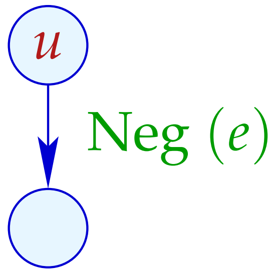
Transformation 4: Removal of Dead Code

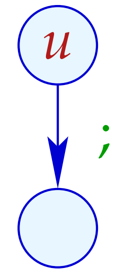




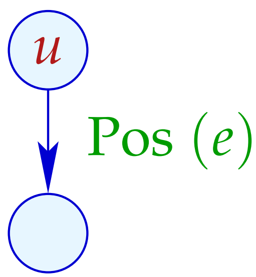
Transformation 4 (cont.): Removal of Dead Code

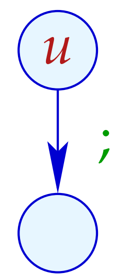
$$\bot \neq \mathcal{D}[u] = D$$
$$[\![e]\!]^\sharp\, D = 0$$





$$\bot \neq \mathcal{D}[u] = D$$
$$[\![e]\!]^\sharp\, D \not\in \{0, \top\}$$





Transformation 4 (cont.): Simplified Expressions

$$\bot \neq \mathcal{D}[u] = D$$
$$[\![e]\!]^\sharp\, D = c$$