The Effects of Edges:

$$
\begin{array}{lcl}
[\![(\_,;,\_)]\!]^\sharp\,(D,M) & = & (D,M) \\
[\![(\_,\mathrm{Pos}(e),\_)]\!]^\sharp\,(D,M) & = & (D,M) \\
[\![(\_,x=y;,\_)]\!]^\sharp\,(D,M) & = & (D \oplus \{x \mapsto D\,y\}, M) \\
[\![(\_,x=e;,\_)]\!]^\sharp\,(D,M) & = & (D \oplus \{x \mapsto \emptyset\}, M) \qquad , \qquad e \notin \mathit{Vars} \\
\\
[\![(u,x=\mathsf{new}();,v)]\!]^\sharp\,(D,M) & = & (D \oplus \{x \mapsto \{(u,v)\}\}, M) \\
[\![(\_,x=y[e];,\_)]\!]^\sharp\,(D,M) & = & (D \oplus \{x \mapsto \bigcup\{M(f) \mid f \in D\,y\}\}, M) \\
[\![(\_,y[e_1]=x;,\_)]\!]^\sharp\,(D,M) & = & (D, M \oplus \{f \mapsto (M\,f \cup D\,x) \mid f \in D\,y\})
\end{array}
$$

Warning:

- The value Null has been ignored. Dereferencing of Null or negative indices are not detected :-(
- Destructive updates are only possible for variables, not for blocks in storage!

  $\Longrightarrow$ no information, if not all block entries are initialized before use :-((
- The effects now depend on the edge itself.

  The analysis cannot be proven correct w.r.t. the reference semantics :-(

  In order to prove correctness, we first instrument the concrete semantics with extra information which records where a block has been created.

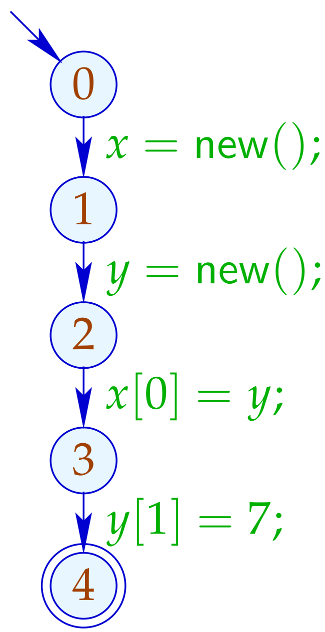
...

- We compute possible points-to information.
- From that, we can extract may-alias information.
- The analysis can be rather expensive — without finding very much :-(
- Separate information for each program point can perhaps be abandoned ??

Alias Analysis 2. Idea:

Compute for each variable and address a value which safely approximates the values at every program point simultaneously !

... in the Simple Example:



x	$\{(0,1)\}$
y	$\{(1,2)\}$
$(0,1)$	$\{(1,2)\}$
$(1,2)$	$\emptyset$

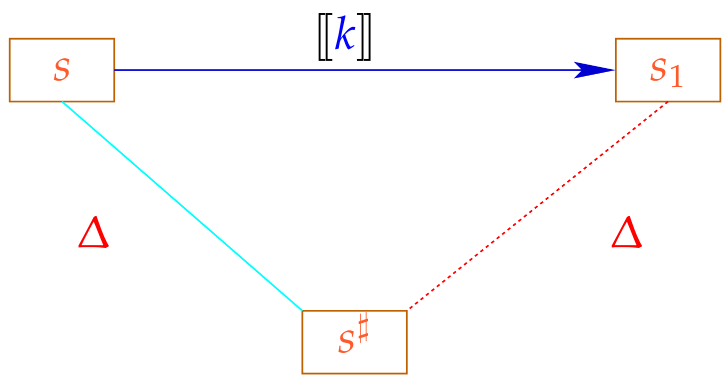
Each edge (u, lab, v) gives rise to constraints:

lab	$Constraint$
$x = y;$	$\mathcal{P}[x] \supseteq \mathcal{P}[y]$
$x = \mathsf{new}();$	$\mathcal{P}[x] \supseteq \{(u, v)\}$
$x = y[e];$	$\mathcal{P}[x] \supseteq \bigcup\{\mathcal{P}[f] \mid f \in \mathcal{P}[y]\}$
$y[e_1] = x;$	$\mathcal{P}[f] \supseteq (f \in \mathcal{P}[y])\, ?\, \mathcal{P}[x] \,:\, \emptyset$
	for all $f \in Addr^\sharp$

Other edges have no effect :-)

Discussion:

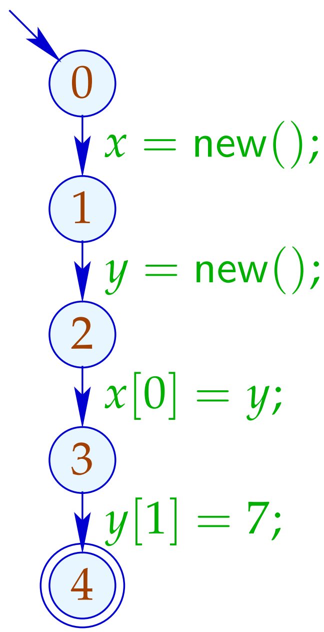
- The resulting constraint system has size $\mathcal{O}(k \cdot n)$ for k abstract addresses and n edges :-(
- The number of necessary iterations is $\mathcal{O}(k)$ ...
- The computed information is perhaps still too zu precise !!?
- In order to prove correctness of a solution $s^\sharp \in States^\sharp$ we show:



Alias Analysis 3. Idea:

Determine one equivalence relation $\equiv$ on variables x and memory accesses $y[\,]$ with $s_1 \equiv s_2$ whenever s_1, s_2 may contain the same address at some u_1, u_2

... in the Simple Example:



$$\equiv \;=\; \{\{x\},\; \{y, x[\,]\},\; \{y[\,]\}\}$$

Discussion:

→ We compute a single information fo the whole program.

→ The computation of this information maintains partitions $\pi = \{P_1, \ldots, P_m\}$:-)

→ Individual sets P_i are identified by means of representatives $p_i \in P_i$.

→ The operations on a partition π are:

$$
\begin{array}{lcll}
\mathsf{find}\,(\pi, p) & = & p_i & \text{if } p \in P_i \\
 & // & \text{returns the representative} & \\
\mathsf{union}\,(\pi, p_{i_1}, p_{i_2}) & = & \{P_{i_1} \cup P_{i_2}\} \cup \{P_j \mid i_1 \neq j \neq i_2\} & \\
 & // & \text{unions the represented classes} &
\end{array}
$$

→ If $x_1, x_2 \in \mathit{Vars}$ are equivalent, then also $x_1[\,]$ and $x_2[\,]$ must be equivalent :-)

→ If $P_i \cap \mathit{Vars} \neq \emptyset$, then we choose $p_i \in \mathit{Vars}$. Then we can apply union recursively :

$$
\begin{array}{lcl}
\mathsf{union}^*\,(\pi, q_1, q_2) & = & \mathsf{let}\;\; p_{i_1} \;=\; \mathsf{find}\,(\pi, q_1) \\
 & & \phantom{\mathsf{let}}\;\; p_{i_2} \;=\; \mathsf{find}\,(\pi, q_2) \\
 & & \mathsf{in}\;\; \mathsf{if}\; p_{i_1} == p_{i_2}\; \mathsf{then}\; \pi \\
 & & \quad \mathsf{else}\;\; \mathsf{let}\; \pi \;=\; \mathsf{union}\,(\pi, p_{i_1}, p_{i_2}) \\
 & & \qquad \mathsf{in}\;\; \mathsf{if}\; p_{i_1}, p_{i_2} \in \mathit{Vars}\; \mathsf{then} \\
 & & \qquad\quad \mathsf{union}^*\,(\pi, p_{i_1}[\,], p_{i_2}[\,])
\end{array}
$$

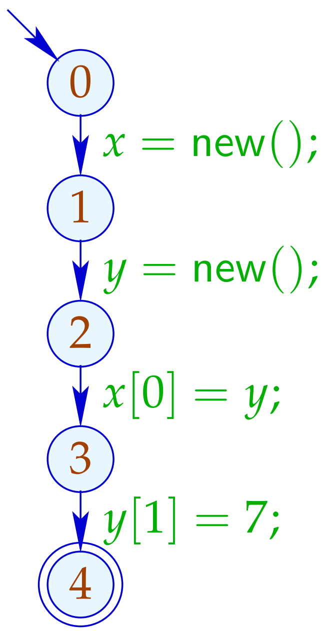
The analysis iterates over all edges once:

$$\pi = \{\{x\}, \{x[\,]\} \mid x \in Vars\};$$
$$\textsf{forall} \quad k = (\_, lab, \_) \quad \textsf{do} \quad \pi = [\![lab]\!]^\sharp\, \pi;$$

where:

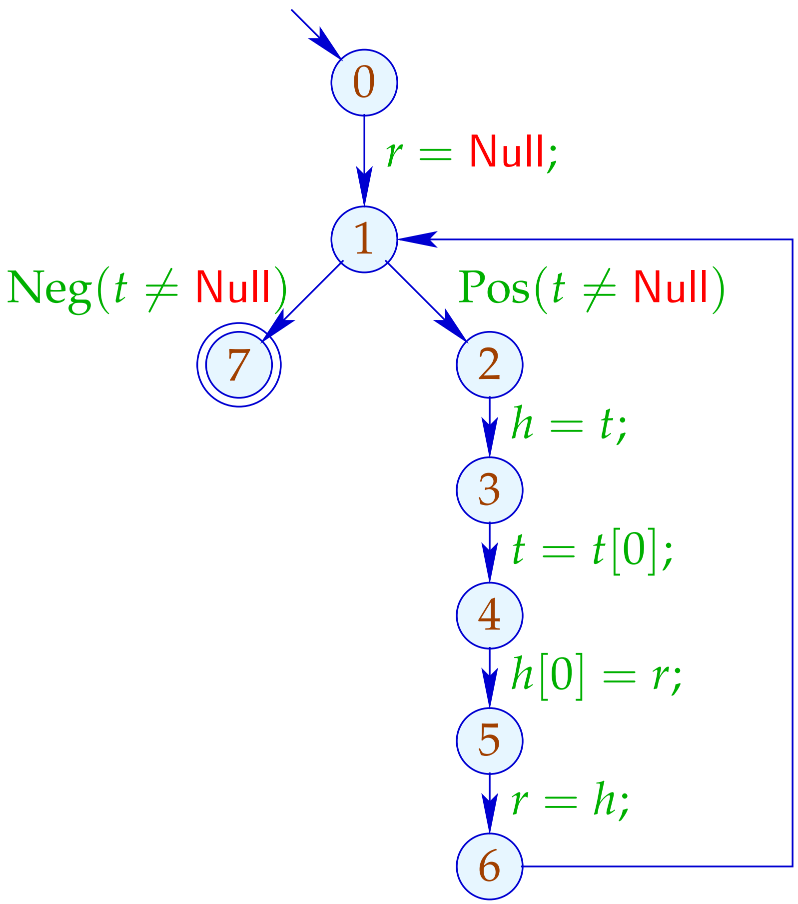
$$\begin{array}{lcll}
[\![x = y;]\!]^\sharp\, \pi & = & \textsf{union}^*\,(\pi, x, y) & \\
[\![x = y[e];]\!]^\sharp\, \pi & = & \textsf{union}^*\,(\pi, x, y[\,]) & \\
[\![y[e] = x;]\!]^\sharp\, \pi & = & \textsf{union}^*\,(\pi, x, y[\,]) & \\
[\![lab]\!]^\sharp\, \pi & = & \pi & \text{otherwise}
\end{array}$$

... in the Simple Example:



	$\{\{x\},\{y\},\{x[\,]\},\{y[\,]\}\}$
$(0,1)$	$\{\{x\},\{y\},\{x[\,]\},\{y[\,]\}\}$
$(1,2)$	$\{\{x\},\{y\},\{x[\,]\},\{y[\,]\}\}$
$(2,3)$	$\{\{x\},\boxed{\{y,x[\,]\}},\{y[\,]\}\}$
$(3,4)$	$\{\{x\},\{y,x[\,]\},\{y[\,]\}\}$

... in the More Complex Example:



	$\{\{h\}, \{r\}, \{t\}, \{h[\,]\}, \{t[\,]\}\}$
$(2, 3)$	$\{\boxed{\{h, t\}}, \{r\}, \boxed{\{h[\,], t[\,]\}}\}$
$(3, 4)$	$\{\boxed{\{h, t, h[\,], t[\,]\}}, \{r\}\}$
$(4, 5)$	$\{\boxed{\{h, t, r, h[\,], t[\,]\}}\}$
$(5, 6)$	$\{\{h, t, r, h[\,], t[\,]\}\}$

Warning:

In order to find something, we must assume that variables / addresses always receive a value before they are accessed.

Complexity:

we havve:

$\mathcal{O}(\#\, edges + \#\, Vars)$	calls of	union*
$\mathcal{O}(\#\, edges + \#\, Vars)$	calls of	find
$\mathcal{O}(\#\, Vars)$	calls of	union

$\Longrightarrow$ We require efficient Union-Find data-structure :-)

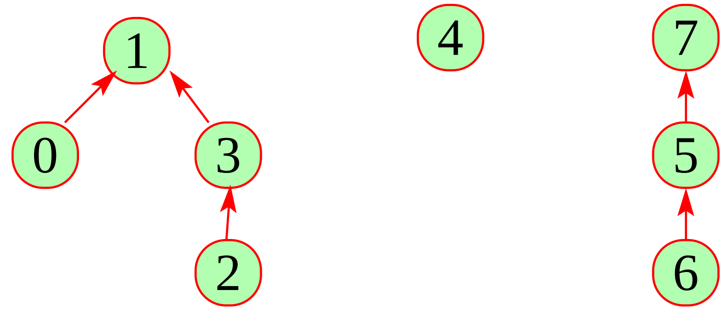
Idea:

Represent partition of U as directed forest:

- For $u \in U$ a reference $F[u]$ to the father is maintained;
- Roots are elements u with $F[u] = u$.

Single trees represent equivalence classes.

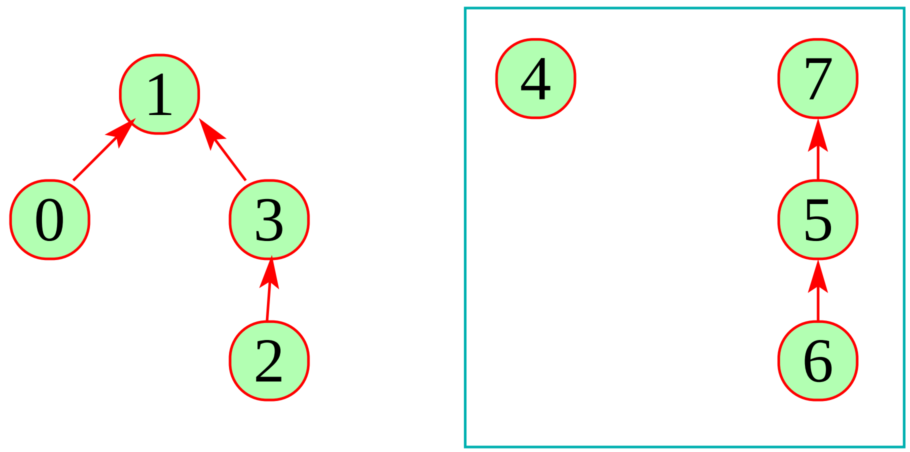
Their roots are their representatives ...



0	1	2	3	4	5	6	7
1	1	3	1	4	7	5	7

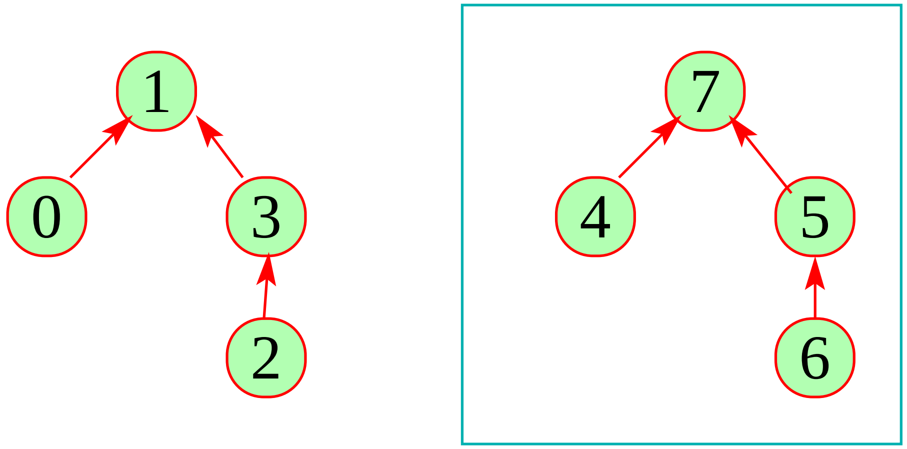
$\longrightarrow$ find (π, u) follows the father references :-)

$\longrightarrow$ union (π, u_1, u_2) re-directs the father reference of one u_i ...



0	1	2	3	4	5	6	7

1	1	3	1	4	7	5	7



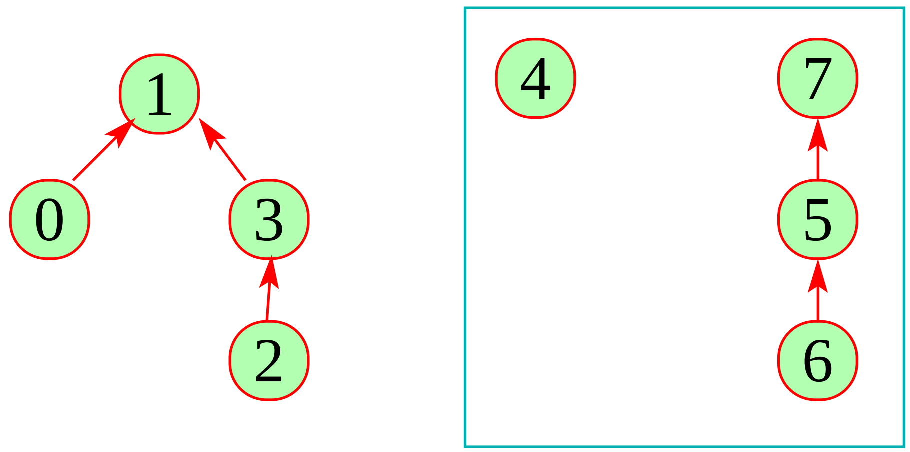
0	1	2	3	4	5	6	7
1	1	3	1	7	7	5	7

The Costs:

union	:	$\mathcal{O}(1)$	:-)
find	:	$\mathcal{O}(depth(\pi))$	:-(

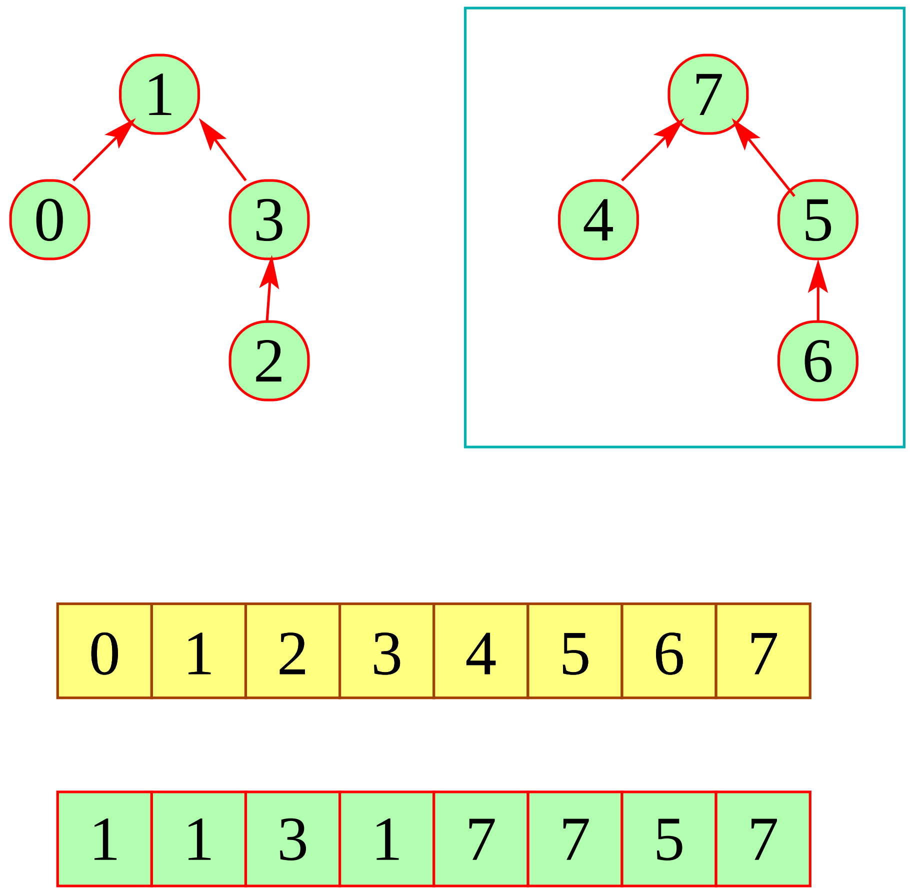
Strategy to Avoid Deep Trees:

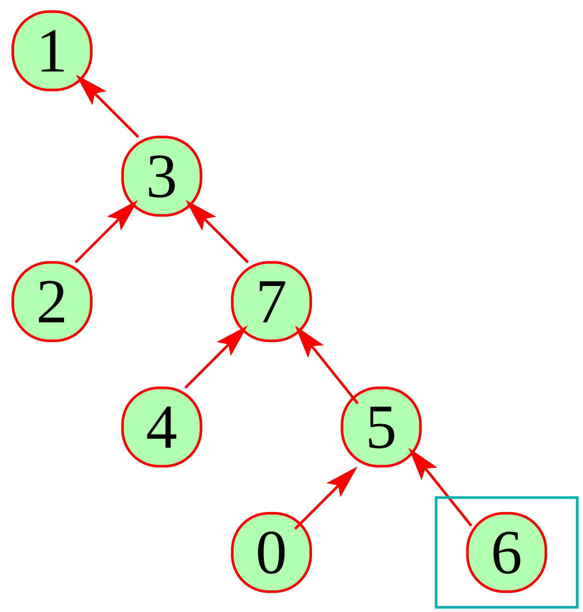
- Put the smaller tree below the bigger !
- Use find to compress paths ...



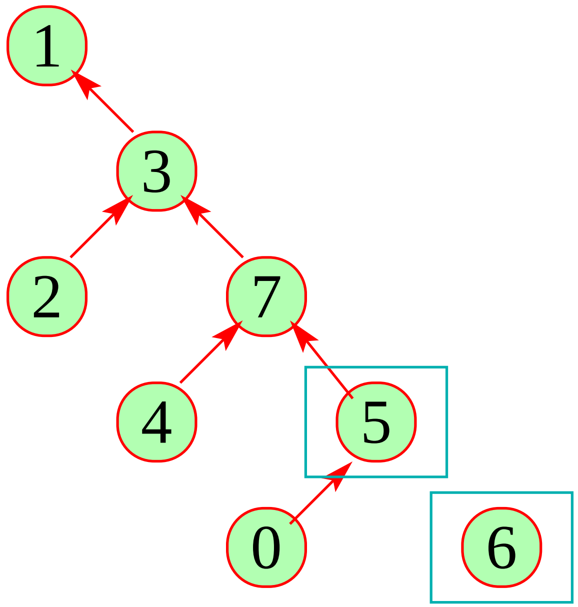
0	1	2	3	4	5	6	7

1	1	3	1	4	7	5	7

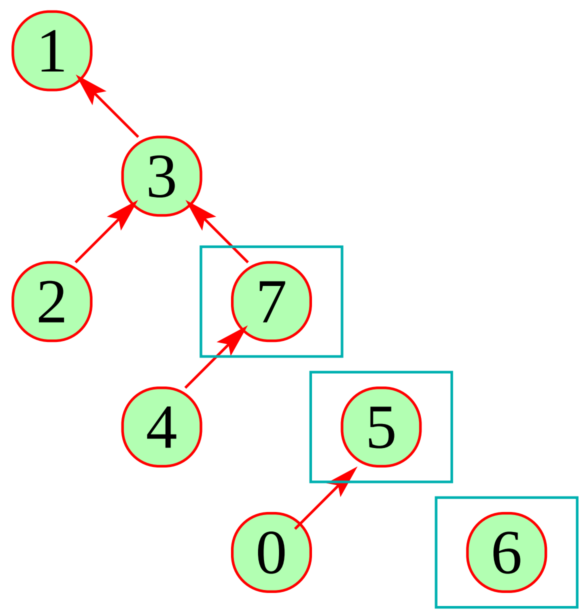

0 1 2 3 4 5 6 7
1 1 3 1 7 7 5 7



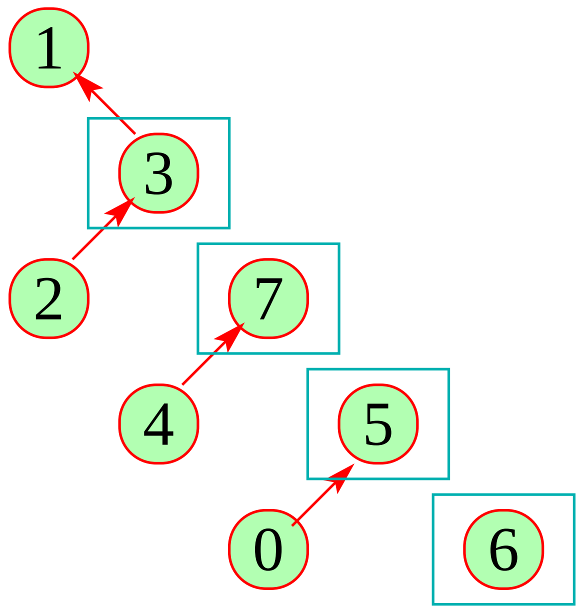
0	1	2	3	4	5	6	7
5	1	3	1	7	7	5	3



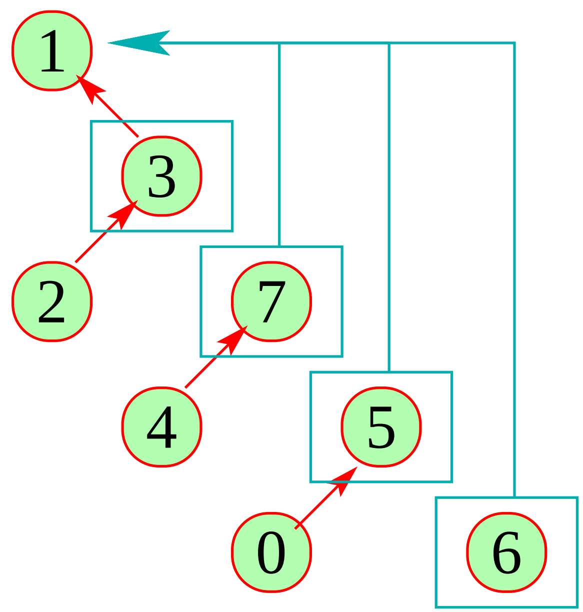
0	1	2	3	4	5	6	7
5	1	3	1	7	7	5	3



0	1	2	3	4	5	6	7
5	1	3	1	7	7	5	3



0	1	2	3	4	5	6	7
5	1	3	1	7	7	5	3



0	1	2	3	4	5	6	7
5	1	3	1	1	7	1	1

Robert Endre Tarjan, Princeton

Note:

- By this data-structure, n union- und m find operations require time $\mathcal{O}(n + m \cdot \alpha(n, n))$

  // α the inverse Ackermann-function :-)

- For our application, we only must modify union such that roots are from *Vars* whenever possible.

- This modification does not increase the asymptotic run-time. :-)

Summary:

The analysis is extremely fast — but may not find very much.

Background 3: Fixpoint Algorithms

Consider: $x_i \sqsupseteq f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n$

Observation:

RR-Iteration is inefficient:

- → We require a complete round in order to detect termination :-(
- → If in some round, the value of just one unknown is changed, then we still re-compute all :-(
- → The practical run-time depends on the ordering on the variables :-(

Idea: Worklist Iteration

If an unknown x_i changes its value, we re-compute all unknowns which depend on x_i . Technically, we require:

- $\longrightarrow$ the lists $Dep\ f_i$ of unknowns which are accessed during evaluation of f_i. From that, we compute the lists:

$$I[x_i] = \{x_j \mid x_i \in Dep\ f_j\}$$

  i.e., a list of all x_j which depend on the value of x_i ;
- $\longrightarrow$ the values $D[x_i]$ of the x_i where initially $D[x_i] = \bot$;
- $\longrightarrow$ a list W of all unknowns whose value must be recomputed ...

The Algorithm:

$$
\begin{array}{l}
W = [x_1, \ldots, x_n]; \\
\textsf{while}\ (W \neq [\,])\ \{ \\
\qquad \begin{array}{lcl}
x_i & = & \textsf{extract}\ W; \\
t & = & f_i\ \textsf{eval}; \\
\end{array} \\
\qquad \textsf{if}\ (t \not\sqsubseteq D[x_i])\ \{ \\
\qquad\qquad \begin{array}{lcl}
D[x_i] & = & D[x_i] \sqcup t; \\
W & = & \textsf{append}\ I[x_i]\ W; \\
\end{array} \\
\qquad \} \\
\}
\end{array}
$$

where :

$$
\mathit{eval}\ x_j \quad = \quad D[x_j]
$$

Example:

$$\begin{array}{lll} x_1 & \supseteq & \{a\} \cup x_3 \\ x_2 & \supseteq & x_3 \cap \{a, b\} \\ x_3 & \supseteq & x_1 \cup \{c\} \end{array}$$

	I
x_1	$\{x_3\}$
x_2	$\emptyset$
x_3	$\{x_1, x_2\}$

Example:

$$
\begin{aligned}
x_1 &\supseteq \{a\} \cup x_3 \\
x_2 &\supseteq x_3 \cap \{a, b\} \\
x_3 &\supseteq x_1 \cup \{c\}
\end{aligned}
$$

	I
x_1	$\{x_3\}$
x_2	$\emptyset$
x_3	$\{x_1, x_2\}$

$D[x_1]$	$D[x_2]$	$D[x_3]$	W
$\emptyset$	$\emptyset$	$\emptyset$	$\boxed{x_1}, x_2, x_3$
$\{a\}$	$\emptyset$	$\emptyset$	$\boxed{x_2}, x_3$
$\{a\}$	$\emptyset$	$\emptyset$	$\boxed{x_3}$
$\{a\}$	$\emptyset$	$\{a, c\}$	$\boxed{x_1}, x_2$
$\{a, c\}$	$\emptyset$	$\{a, c\}$	$\boxed{x_3}, x_2$
$\{a, c\}$	$\emptyset$	$\{a, c\}$	$\boxed{x_2}$
$\{a, c\}$	$\{a\}$	$\{a, c\}$	$[\,]$

Theorem

Let $x_i \sqsupseteq f_i\,(x_1, \ldots, x_n)\;,\quad i = 1, \ldots, n$ denote a constraint system over the complete lattice $\mathbb{D}$ of hight $h > 0$.

(1) The algorithm terminates after at most $h \cdot N$ evaluations of right-hand sides where

$$N = \sum_{i=1}^{n} (1 + \#\,(\mathit{Dep}\; f_i)) \qquad // \quad \text{size of the system} \quad \text{:-)}$$

(2) The algorithm returns a solution.
If all f_i are monotonic, it returns the least one.