Problem:

- The definition $x = e;$ $(x \notin Vars_e)$ may only be moved to an edge where e is safe ;-)
- The definition must still be available for uses of x ;-)

$$\Longrightarrow$$

We define an analysis which maximally delays computations:

$$\begin{array}{lll}
[\![;]\!]^\sharp\, D & = & D \\
[\![x = e;]\!]^\sharp\, D & = & \begin{cases} D \backslash (Use_e \cup Def_x) \cup \{x = e;\} & \text{if} \quad x \notin Vars_e \\ D \backslash (Use_e \cup Def_x) & \text{if} \quad x \in Vars_e \end{cases}
\end{array}$$

... where:

$$\begin{array}{rcl} Use_e & = & \{y = e'; \mid y \in Vars_e\} \\ Def_x & = & \{y = e'; \mid y \equiv x \vee x \in Vars_{e'}\} \end{array}$$

... where:

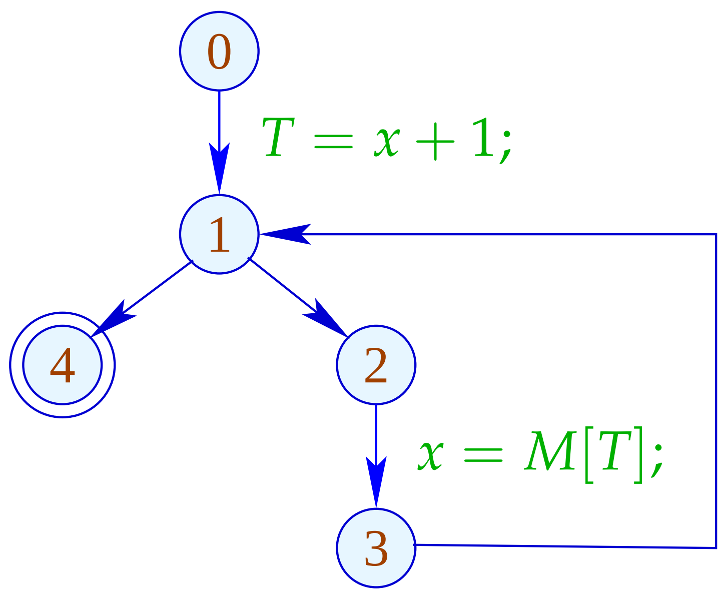
$$\begin{array}{lll} Use_e & = & \{y = e'; \mid y \in Vars_e\} \\ Def_x & = & \{y = e'; \mid y \equiv x \vee x \in Vars_{e'}\} \end{array}$$

For the remaining edges, we define:

$$\begin{array}{lllll} [\![x = M[e];]\!]^\sharp\, D & = & D \backslash (Use_e \cup Def_x) \\ [\![M[e_1] = e_2;]\!]^\sharp\, D & = & D \backslash (Use_{e_1} \cup Use_{e_2}) \\ [\![\mathsf{Pos}(e)]\!]^\sharp\, D & = & [\![\mathsf{Neg}(e)]\!]^\sharp\, D & = & D \backslash Use_e \end{array}$$

Warning:

We may move $y = e;$ beyond a join only if $y = e;$ can be delayed along all joining edges:



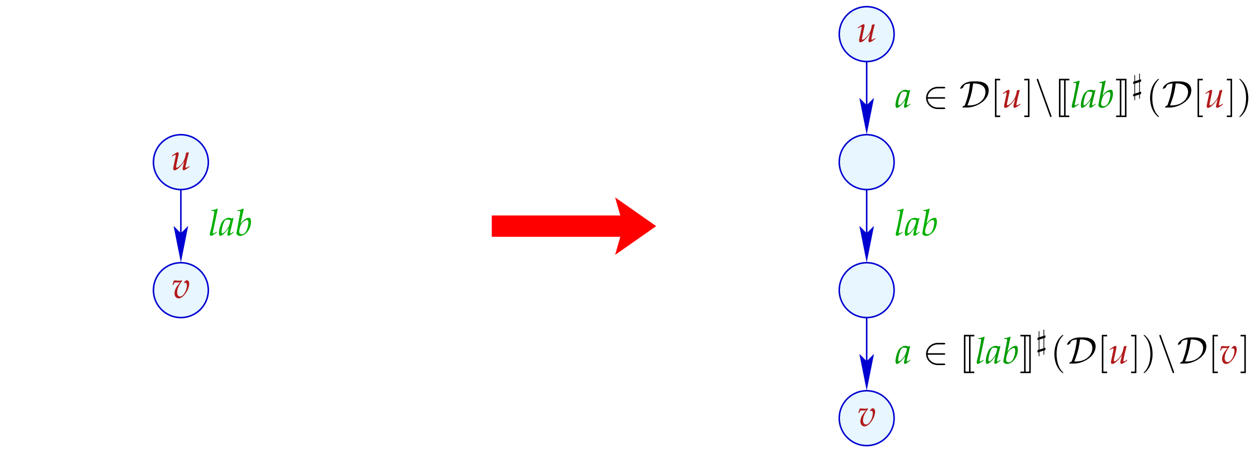
Here, $T = x + 1;$ cannot be moved beyond 1 !!!

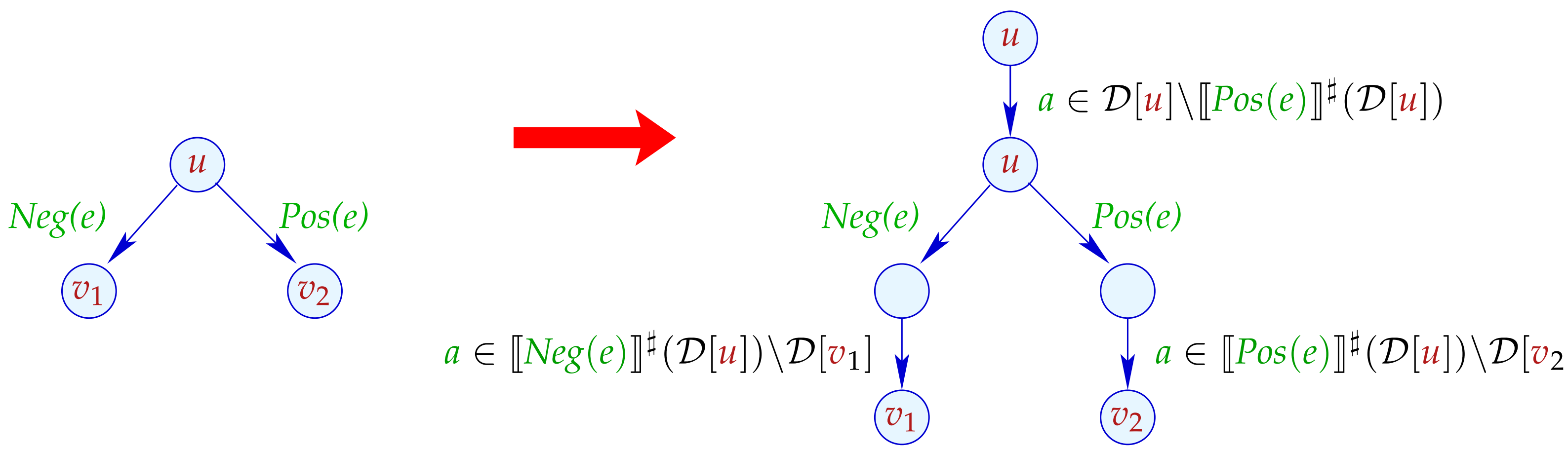
We conclude:

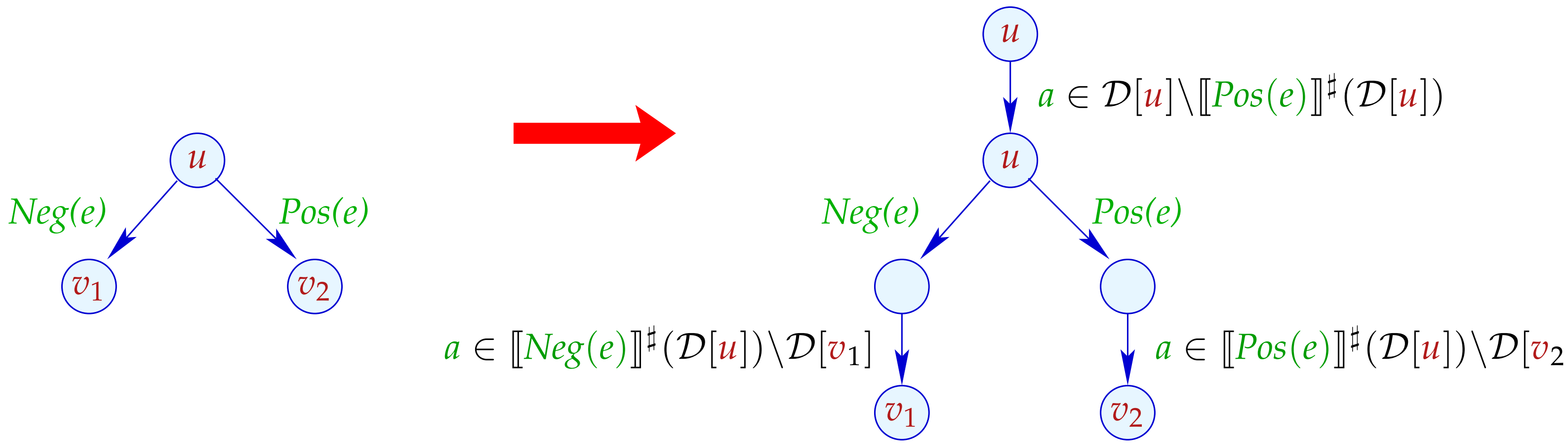
- The partial ordering of the lattice for delayability is given by "$\supseteq$".
- At program start: $D_0 = \emptyset$.

  Therefore, the sets $\mathcal{D}[u]$ of at u delayable assignments can be computed by solving a system of constraints.
- We delay only assignments a where $a\,a$ has the same effect as a alone.
- The extra insertions render the original assignments as assignments to dead variables ...

Transformation 7:



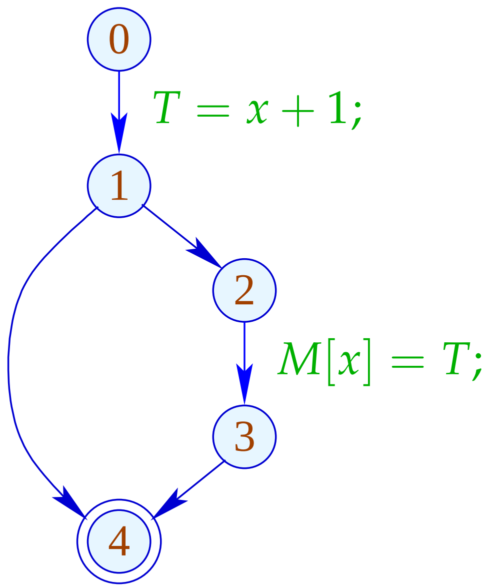




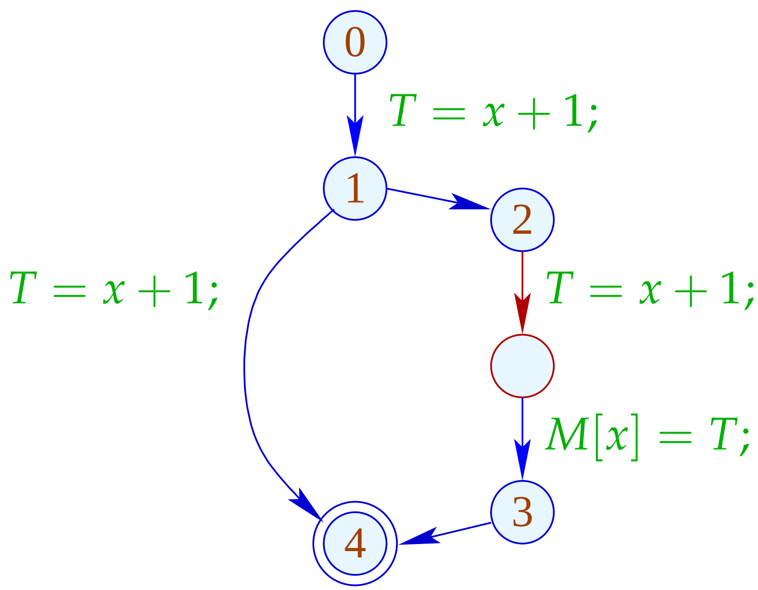
Note:

Transformation T7 is only meaningful, if we subsequently eliminate assignments to dead variables by means of transformation T2 :-)

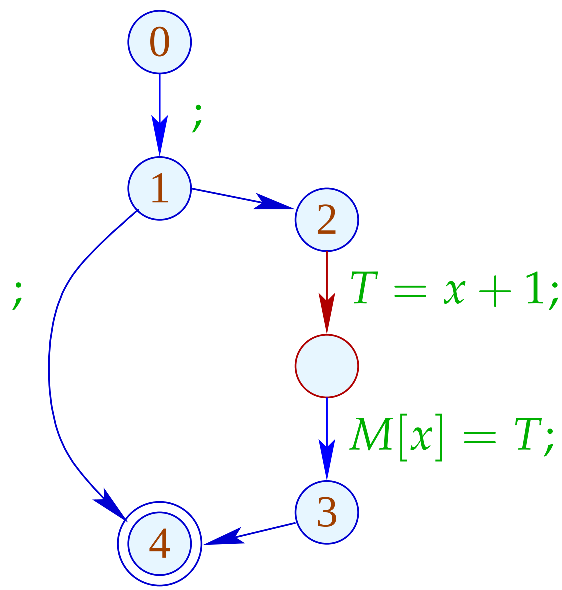
In the example, the partially dead code is eliminated:



	$\mathcal{D}$
0	$\emptyset$
1	$\{T = x + 1;\}$
2	$\{T = x + 1;\}$
3	$\emptyset$
4	$\emptyset$



	$\mathcal{D}$
0	$\emptyset$
1	$\{T = x + 1;\}$
2	$\{T = x + 1;\}$
3	$\emptyset$
4	$\emptyset$



	$\mathcal{L}$
0	$\{x\}$
1	$\{x\}$
2	$\{x\}$
$2'$	$\{x, T\}$
3	$\emptyset$
4	$\emptyset$

Remarks:

- After $T7$, all original assignments $y = e;$ with $y \notin Vars_e$ are assignments to dead variables and thus can always be eliminated :-)
- By this, it can be proven that the transformation is guaranteed to be non-degradating efficiency of the code :-))
- Similar to the elimination of partial redundancies, the transformation can be repeated :-}

Conclusion:

→ The design of a meaningful optimization is non-trivial.

→ Many transformations are advantageous only in connection with other optimizations :-)

→ The ordering of applied optimizations matters !!

→ Some optimizations can be iterated !!!

... a meaningful ordering:

T4	Constant Propagation Interval Analysis Alias Analysis
T6	Loop Rotation
T1, T3, T2	Available Expressions
T2	Dead Variables
T7, T2	Partially Dead Code
T5, T3, T2	Partially Redundant Code

2 Replacing Expensive Operations by Cheaper Ones

2.1 Reduction of Strength

(1) Evaluation of Polynomials

$$f(x) \;=\; a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \ldots + a_1 \cdot x + a_0$$

	Multiplications	Additions
naive	$\frac{1}{2}n(n+1)$	n
re-use	$2n-1$	n
Horner-Scheme	n	n

Idea:

$$f(x) = (\ldots((a_n \cdot x + a_{n-1}) \cdot x + a_{n-2}) \ldots) \cdot x + a_0$$

(2) Tabulation of a polynomial $f(x)$ of degree n :

$\longrightarrow$ To recompute $f(x)$ for every argument x is too expensive :-)

$\longrightarrow$ Luckily, the n-th differences are constant !!!

Example: $f(x) = 3x^3 - 5x^2 + 4x + 13$

n	$f(n)$	Δ	Δ^2	Δ^3
0	13	2	8	$\boxed{18}$
1	15	10	$\boxed{26}$	
2	25	$\boxed{36}$		
3	$\boxed{61}$			
4	...			

Here, the n-th difference is always

$$\Delta_h^n(f) = n! \cdot a_n \cdot h^n \qquad (h \text{ step width})$$

Costs:

- n times evaluation of f ;
- $\frac{1}{2} \cdot (n-1) \cdot n$ subtractions to determine the Δ^k ;
- $2n-2$ multiplications for computing $\Delta_h^n(f)$;
- n additions for every further value :-)

$$\Longrightarrow$$

Number of multiplications only depends on n :-))

Simple Case: $f(x) = a_1 \cdot x + a_0$

- ... naturally occurs in many numerical loops :-)
- The first differences are already constant:

$$f(x+h) - f(x) = a_1 \cdot h$$

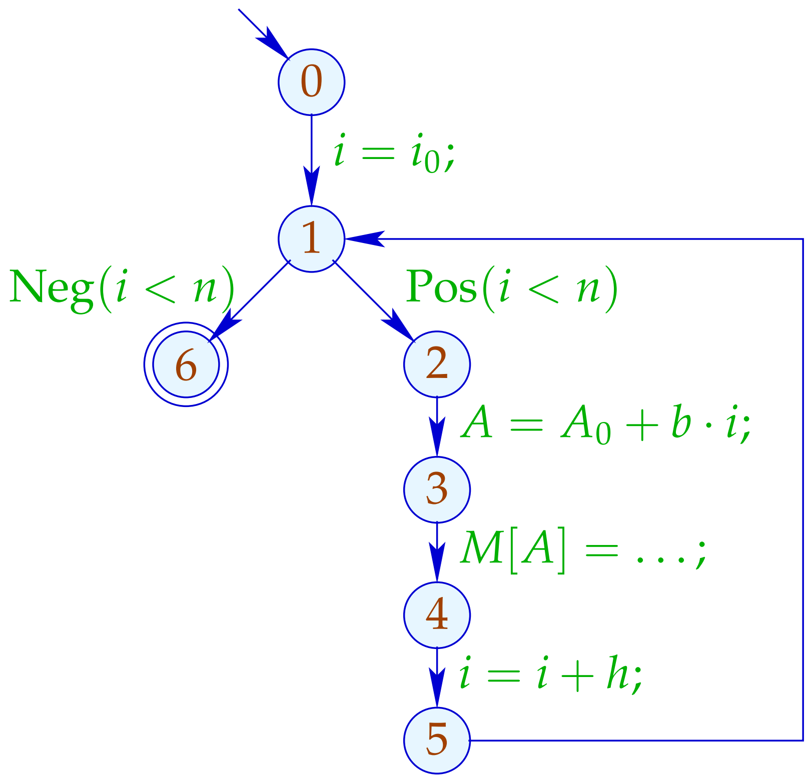
- Instead of the sequence: $y_i = f(x_0 + i \cdot h), \quad i \geq 0$

  we compute: $y_0 = f(x_0), \quad \Delta = a_1 \cdot h$

  $y_i = y_{i-1} + \Delta, \quad i > 0$

Example:

$$\text{for } (i = i_0; i < n; i = i + h) \ \{$$
$$A = A_0 + b \cdot i;$$
$$M[A] = \ldots;$$
$$\}$$



... or, after loop rotation:

$i = i_0;$

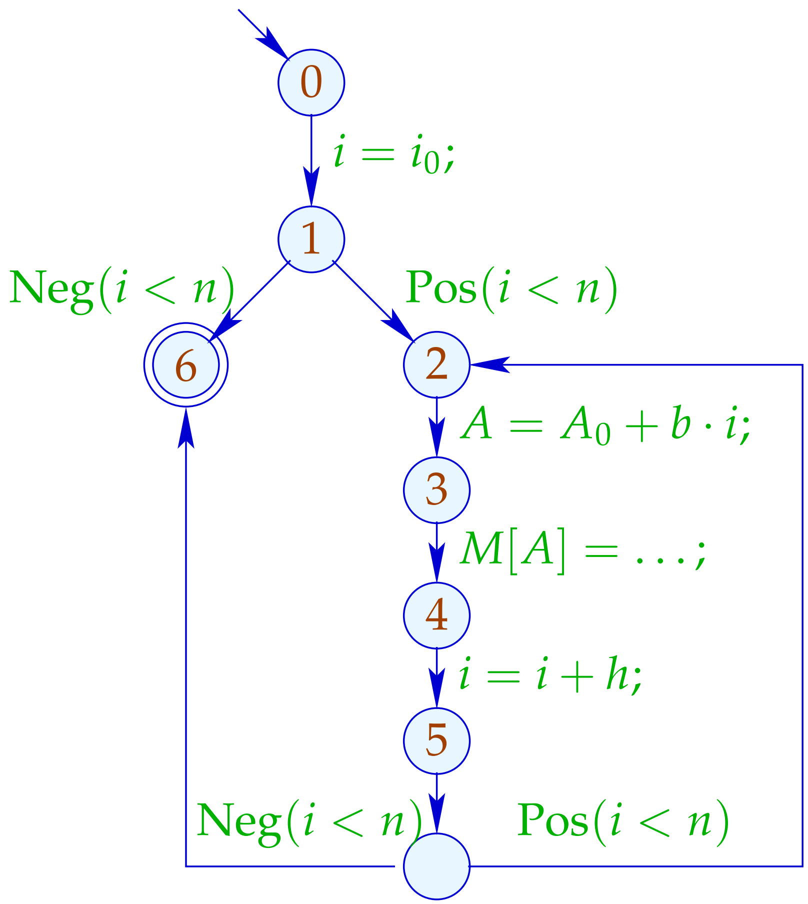
if $(i < n)$ do $\{$

$A = A_0 + b \cdot i;$

$M[A] = \ldots;$

$i = i + h;$

$\}$ while $(i < n);$



... and reduction of strength:

$i = i_0;$

if $(i < n)$ {

  $\Delta = b \cdot h;$

  $A = A_0 + b \cdot i_0;$

  do {

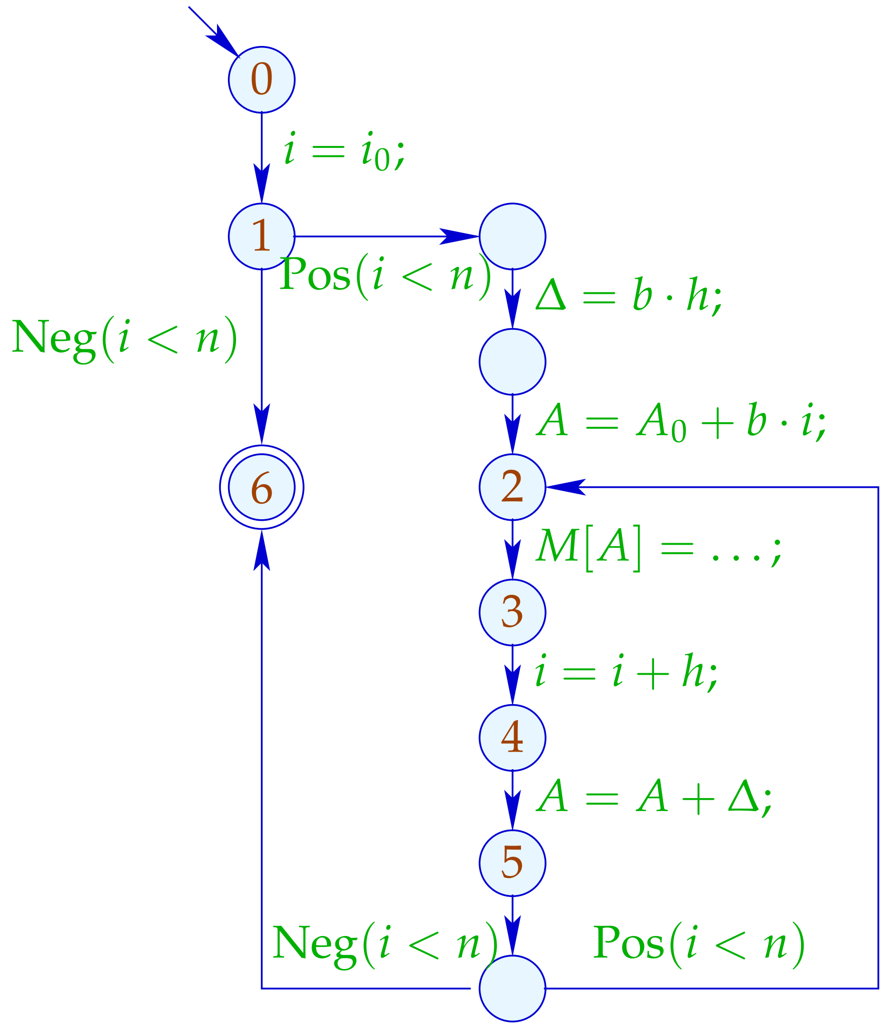
    $M[A] = \ldots;$

    $i = i + h;$

    $A = A + \Delta;$

  } while $(i < n);$

}



Warning:

- The values b, h, A_0 must not change their values during the loop.
- i, A may be modified at exactly one position in the loop :-(
- One may try to eliminate the variable i altogether :

  $\longrightarrow$ i may not be used else-where.

  $\longrightarrow$ The initialization must be transformed into:
  $A = A_0 + b \cdot i_0$.

  $\longrightarrow$ The loop condition $i < n$ must be transformed into:
  $A < N$ for $N = A_0 + b \cdot n$.

  $\longrightarrow$ b must always be different from zero !!!

Approach:

Identify

... loops;

... iteration variables;

... constants;

... the matching use structures.

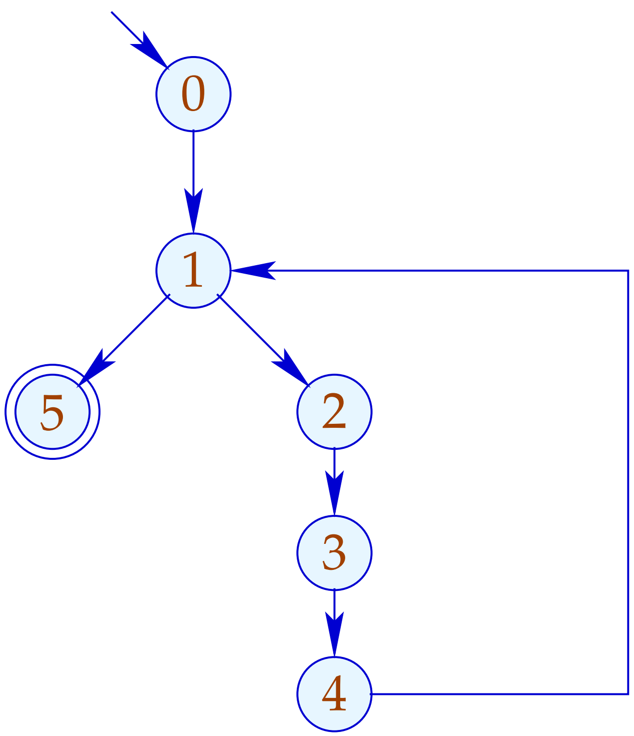
Loops:

... are identified through the node v with back edge $(\_, \_, v)$:-)

For the sub-graph G_v of the cfg on $\{w \mid v \Rightarrow w\}$, we define:

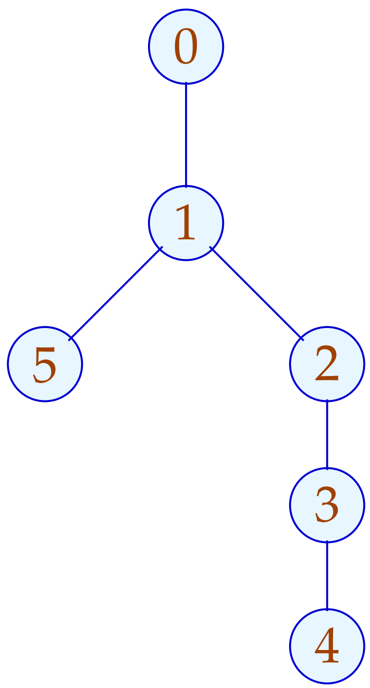
$$\mathsf{Loop}[v] \quad = \quad \{w \mid w \longrightarrow^* v \;\; \text{in} \;\; G_v\}$$

Example:



	$\mathcal{P}$
0	$\{0\}$
1	$\{0,1\}$
2	$\{0,1,2\}$
3	$\{0,1,2,3\}$
4	$\{0,1,2,3,4\}$
5	$\{0,1,5\}$

Example:



	$\mathcal{P}$
0	$\{0\}$
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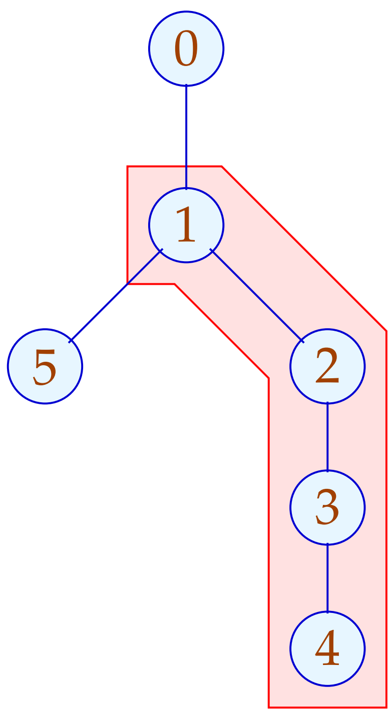
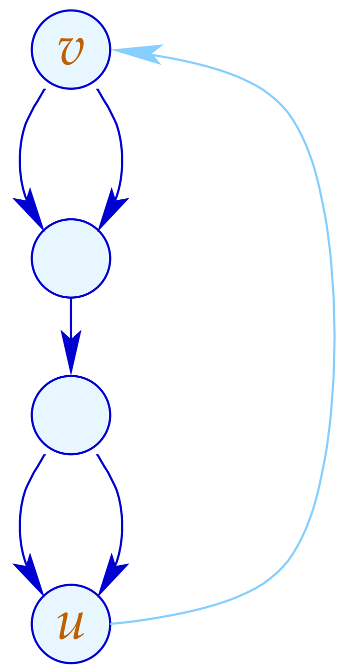
3	$\{0,1,2,3\}$
4	$\{0,1,2,3,4\}$
5	$\{0,1,5\}$

Example:



	$\mathcal{P}$
0	$\{0\}$
1	$\{0, 1\}$
2	$\{0, 1, 2\}$
3	$\{0, 1, 2, 3\}$
4	$\{0, 1, 2, 3, 4\}$
5	$\{0, 1, 5\}$

We are interested in edges which during each iteration are executed exactly once:



Graph-theoretically, this is not easily expressible :-(

Edges k could be selected such that:

- the sub-graph $G = \mathsf{Loop}[v] \backslash \{(\_, \_, v)\}$ is connected;
- the graph $G \backslash \{k\}$ is split into two unconnected sub-graphs.

Edges k could be selected such that:

- the sub-graph $G = \mathsf{Loop}[v] \backslash \{(\_, \_, v)\}$ is connected;
- the graph $G \backslash \{k\}$ is split into two unconnected sub-graphs.

On the level of source programs, this is trivial:

$$\begin{array}{l} \mathsf{do}\ \ \{\ \ s_1 \ldots s_k \\ \quad\ \ \}\ \ \mathsf{while}\ \ (e); \end{array}$$

The desired assignments must be among the s_i :-)

Iteration Variable:

i is an iteration variable if the only definition of i inside the loop occurs at an edge which separates the body and is of the form:

$$i = i + h;$$

for some loop constant h .

A loop constant is simply a constant (e.g., 42), or slightly more libaral, an expression which only depends on variables which are not modified during the loop :-)

(3) Differences for Sets

Consider the fixpoint computation:

$$\begin{array}{l} x = \emptyset; \\ \mathsf{for}\ \ (t = F\,x; t \not\subseteq x; \boxed{t = F\,x;}) \\ \qquad\qquad x = x \cup t; \end{array}$$

If F is distributive, it could be replaced by:

$$\begin{array}{l} x = \emptyset; \\ \mathsf{for}\ \ (\Delta = F\,x; \Delta \neq \emptyset; \boxed{\Delta = (F\,\Delta) \setminus x;}) \\ \qquad\qquad x = x \cup \Delta; \end{array}$$

The function F must only be computed for the smaller sets Δ

:-) semi-naive iteration

$$
\begin{array}{lllllll}
\text{Instead of the sequence:} & \emptyset & \subseteq & F(\emptyset) & \subseteq & F^2(\emptyset) & \subseteq \quad \ldots \\
\text{we compute:} & \Delta_1 & \cup & \Delta_2 & \cup & \ldots & \\
\text{where:} & \Delta_{i+1} & = & F(F^i(\emptyset)) \backslash F^i(\emptyset) & & & \\
 & & = & F(\Delta_i) \backslash (\Delta_1 \cup \ldots \cup \Delta_i) & \text{with } \Delta_0 = \emptyset & &
\end{array}
$$

Assume that the costs of $F\ x$ is $1 + \#x$.

Then the costs may sum up to:

naive	$1 + 2 + \ldots + n + n$	$=$	$\frac{1}{2}n(n+3)$
semi-naive			$2n$

where n is the cardinality of the result.

$\Longrightarrow$ A linear factor is saved :-)

2.2 Peephole Optimization

Idea:

- Slide a small window over the program.
- Optimize agressively inside the window, i.e.,

  → Eliminate redundancies!

  → Replace expensive operations inside the window by cheaper ones!

Examples:

$x = x + 1;$ $\Longrightarrow$ $x{+}{+};$

// given that there is a specific increment instruction :-)

$z = y - a + a;$ $\Longrightarrow$ $z = y;$

// algebraic simplifications :-)

$x = x;$ $\Longrightarrow$ $;$

$x = 0;$ $\Longrightarrow$ $x = x \oplus x;$

$x = 2 \cdot x;$ $\Longrightarrow$ $x = x + x;$