Generalization: Static Single Assignment Form

We proceed in two phases:

**Step 1:**
Transform the program such that each program point \( v \) is reached by at most one definition of a variable \( x \) which is live at \( v \).

**Step 2:**
- Introduce a separate variant \( x_i \) for every occurrence of a definition of a variable \( x \)!
- Replace every use of \( x \) with the use of the reaching variant \( x_h \) ...
Implementing Step 1:

- Determine for every program point the set of reaching definitions.
- If the join point $v$ is reached by more than one definition for the same variable $x$ which is live at program point $v$, insert definitions $x = x$; at the end of each incoming edge.
Example

Reaching Definitions

<table>
<thead>
<tr>
<th></th>
<th>(\mathcal{R})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(\langle x, 0\rangle, \langle y, 0\rangle)</td>
</tr>
<tr>
<td>1</td>
<td>(\langle x, 1\rangle, \langle y, 0\rangle)</td>
</tr>
<tr>
<td>2</td>
<td>(\langle x, 1\rangle, \langle x, 5\rangle, \langle y, 2\rangle, \langle y, 4\rangle)</td>
</tr>
<tr>
<td>3</td>
<td>(\langle x, 1\rangle, \langle x, 5\rangle, \langle y, 2\rangle, \langle y, 4\rangle)</td>
</tr>
<tr>
<td>4</td>
<td>(\langle x, 1\rangle, \langle x, 5\rangle, \langle y, 4\rangle)</td>
</tr>
<tr>
<td>5</td>
<td>(\langle x, 5\rangle, \langle y, 4\rangle)</td>
</tr>
<tr>
<td>6</td>
<td>(\langle x, 1\rangle, \langle x, 5\rangle, \langle y, 2\rangle, \langle y, 4\rangle)</td>
</tr>
<tr>
<td>7</td>
<td>(\langle x, 1\rangle, \langle x, 5\rangle, \langle y, 2\rangle, \langle y, 4\rangle)</td>
</tr>
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</table>

\[x = M[I];\]

\[y = 1;\]

\[y = x \ast y;\]

\[x = x - 1;\]

\[M[R] = y;\]
Example

\[ x = M[I]; \]
\[ y = 1; \]
\[ y = x \ast y; \]
\[ x = x - 1; \]
\[ M[R] = y; \]

Neg\((x > 1)\)

\[ \psi \equiv x = x \mid y = y \]

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Reaching Definitions

The complete lattice $\mathbb{R}$ for this analysis is given by:

$$\mathbb{R} = 2^{\text{Defs}}$$

where

$$\text{Defs} = \text{Vars} \times \text{Nodes} \quad \text{Defs}(x) = \{x\} \times \text{Nodes}$$

Then:

$$\llbracket (_{}, x = r; , v) \rrbracket ^\# \mathbb{R} = R \setminus \text{Defs}(x) \cup \{\langle x, v \rangle \}$$
$$\llbracket (_{}, x = x \mid x \in L, v) \rrbracket ^\# \mathbb{R} = R \setminus \bigcup_{x \in L} \text{Defs}(x) \cup \{\langle x, v \rangle \mid x \in L\}$$

The ordering on $\mathbb{R}$ is given by subset inclusion $\subseteq$ where the value at program start is given by $R_0 = \{\langle x, \text{start} \rangle \mid x \in \text{Vars}\}$. 
The Transformation SSA, Step 1:

where $k \geq 2$.

The label $\psi$ of the new in-going edges for $v$ is given by:

$$\psi \equiv \{ x = x \mid x \in \mathcal{L}[v], \#(\mathcal{R}[v] \cap \text{Defs}(x)) > 1 \}$$
If the node $v$ is the start point of the program, we add auxiliary edges whenever there are further ingoing edges into $v$:

**The Transformation SSA, Step 1 (cont.):**

![Diagram](image.png)

where $k \geq 1$ and $\psi$ of the new in-going edges for $v$ is given by:

$$\psi \equiv \{ x = x \mid x \in L[v], \#(R[v] \cap \text{Defs}(x)) > 1 \}$$
Discussion

- Program start is interpreted as (the end point of) a definition of every variable $x$ :-)
- At some edges, parallel definitions $\psi$ are introduced !
- Some of them may be useless :-(
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• At some edges, parallel definitions \( \psi \) are introduced !

• Some of them may be useless :-(

Improvement:

• We introduce assignments \( x = x \) before \( v \) only if the sets of reaching definitions for \( x \) at incoming edges of \( v \) differ !

• This introduction is repeated until every \( v \) is reached by exactly one definition for each variable live at \( v \).
Theorem

Assume that every program point in the controlflow graph is reachable from \texttt{start} and that every left-hand side of a definition is live. Then:

1. The algorithm for inserting definitions \( x = x \) terminates after at most \( n \cdot (m + 1) \) rounds were \( m \) is the number of program points with more than one in-going edges and \( n \) is the number of variables.

2. After termination, for every program point \( u \), the set \( \mathcal{R}[u] \) has exactly one definition for every variable \( x \) which is live at \( u \).
Discussion

The efficiency crucially depends on the number of iterations. If the cfg is well-structured, it terminates already after one iteration!
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A well-structured cfg can be reduced to a single vertex or edge by:
Discussion (cont.)

• Reducible cfgs are not the exception — but the rule :-) 

• In Java, reducibility is only violated by loops with breaks/continues. 

• If the insertion of definitions does not terminate after $k$ iterations, we may immediately terminate the procedure by inserting definitions $x = x$ before all nodes which are reached by more than one definition of $x$. 

Assume now that every program point $u$ is reached by exactly one definition for each variable which is live at $u$ ...
The Transformation SSA, Step 2:

Each edge \((u, lab, v)\) is replaced with \((u, T_{v, \phi}[lab], v)\) where \(\phi x = x_{u'}\) if \((x, u') \in R[u]\) and:

\[
\begin{align*}
T_{v, \phi}[;] &= ; \\
T_{v, \phi}[Neg(e)] &= Neg(\phi(e)) \\
T_{v, \phi}[Pos(e)] &= Pos(\phi(e)) \\
T_{v, \phi}[x = e] &= x_v = \phi(e) \\
T_{v, \phi}[x = M[e]] &= x_v = M[\phi(e)] \\
T_{v, \phi}[M[e_1] = e_2] &= M[\phi(e_1)] = \phi(e_2) \\
T_{v, \phi}[\{ x = x \mid x \in L \}] &= \{ x_v = \phi(x) \mid x \in L \}
\end{align*}
\]
Remark

The multiple assignments:

\[ pa = x_v^{(1)} = x_{v_1}^{(1)} \mid \ldots \mid x_v^{(k)} = x_{v_k}^{(k)} \]

in the last row are thought to be executed in parallel, i.e.,

\[ \llbracket pa \rrbracket (\rho, \mu) = (\rho \oplus \{ x_v^{(i)} \mapsto \rho(x_v^{(i)}) \mid i = 1, \ldots, k \}, \mu) \]
Example

\[ x_1 = M[I]; \]
\[ y_1 = 1; \]

\[ \psi_1 = x_3 = x_1 \mid y_3 = y_1 \]
\[ \psi_2 = x_3 = x_2 \mid y_3 = y_2 \]
Theorem

Assume that every program point is reachable from start and the program is in SSA form without assignments to dead variables.

Let \( \lambda \) denote the maximal number of simultaneously live variables and \( G \) the interference graph of the program variables. Then:

\[
\lambda = \omega(G) = \chi(G)
\]

where \( \omega(G), \chi(G) \) are the maximal size of a clique in \( G \) and the minimal number of colors for \( G \), respectively.

A minimal coloring of \( G \), i.e., an optimal register allocation can be found in polynomial time.
Discussion

- By the theorem, the number $\lambda$ of required registers can be easily computed :-)

- Thus variables which are to be spilled to memory, can be determined ahead of the subsequent assignment of registers!

- Thus here, we may, e.g., insist on keeping iteration variables from inner loops.
Discussion

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- Thus variables which are to be spilled to memory, can be determined ahead of the subsequent assignment of registers !

- Thus here, we may, e.g., insist on keeping iteration variables from inner loops.

- Clearly, always \( \lambda \leq \omega(G) \leq \chi(G) \) :-)
  Therefore, it suffices to color the interference graph with \( \lambda \) colors.

- Instead, we provide an algorithm which directly operates on the cfg ...

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Observation

- Live ranges of variables in programs in SSA form behave similar to live ranges in basic blocks!
- Consider some DFS spanning tree \( T \) of the CFG with root \( \text{start} \).
- For each variable \( x \), the live range \( L[x] \) forms a tree fragment of \( T \)!
- A tree fragment is a subtree from which some subtrees have been removed ...
Example

\[ x = M[i]; \]
\[ y = 1; \]
\[ \text{Neg}(x > 1) \]
\[ M[a] = y; \]
\[ \text{Pos}(x > 1) \]
\[ y = x \times y; \]
\[ x = x - 1; \]

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Discussion

• Although the example program is not in SSA form, all live ranges still form tree fragments :) 

• The intersection of tree fragments is again a tree fragment !

• A set $C$ of tree fragments forms a clique iff their intersection is non-empty !!!

• The greedy algorithm will find an optimal coloring ...
Proof of the Intersection Property

(1) Assume $I_1 \cap I_2 \neq \emptyset$ and $v_i$ is the root of $I_i$. Then:

$$v_1 \in I_2 \quad \text{or} \quad v_2 \in I_1$$

(2) Let $C$ denote a clique of tree fragments. Then there is an enumeration $C = \{I_1, \ldots, I_r\}$ with roots $v_1, \ldots, v_r$ such that

$$v_i \in I_j \quad \text{for all} \quad j \leq i$$

In particular, $v_r \in I_i$ for all $i$. :-)

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The Greedy Algorithm

forall \( u \in \text{Nodes} \) \( \text{visited}[u] = \text{false} \);
forall \( x \in \mathcal{L}[\text{start}] \) \( \Gamma(x) = \text{extract}(\text{free}) \);
alloc(\text{start});

\textbf{void alloc (Node }u\text{)} \{ \\
\quad \text{visited}[u] = \text{true}; \\
\quad \textbf{forall ((lab, }v\text{) }\in \text{edges}[u]) \\
\quad \quad \textbf{if } (\neg \text{visited}[v]) \{ \\
\quad \quad \quad \textbf{forall } (x \in \mathcal{L}[u] \setminus \mathcal{L}[v]) \text{ insert}(\text{free}, x); \\
\quad \quad \quad \textbf{forall } (x \in \mathcal{L}[v] \setminus \mathcal{L}[u]) \Gamma(x) = \text{extract}(\text{free}); \\
\quad \quad \quad \text{alloc}(v); \\
\quad \quad } \\
\}
Example

read();

x = M[A];

y = x + 1;

Neg (y)  Pos (y)

z = x \cdot x

M[A] = t;

M[A] = z;
Example

$\text{read}();$

$x = M[A];$

$y = x + 1;$

$\text{Neg}(y)$

$\text{Pos}(y)$

$t = -y \cdot y;$

$z = x \cdot x$

$M[A] = t;$

$M[A] = z;$

$\text{read}();$

$R_1 = M[A];$

$R_2 = R_1 + 1;$

$\text{Neg}(R_2)$

$\text{Pos}(R_2)$

$R_1 = -R_2 \cdot R_2;$

$R_1 = R_1 \cdot R_1$

$M[A] = R_1;$

$M[A] = R_1;$
Remark:

- Intersection graphs for tree fragments are also known as cordal graphs ...  
- A cordal graph is an undirected graph where every cycle with more than three nodes contains a cord :-)  
- Cordal graphs are another sub-class of perfect graphs :-)  

- Cheap register allocation comes at a price:  
when transforming into SSA form, we have introduced parallel register-register moves :-(
Problem

The parallel register assignment:

\[ \psi_1 = R_1 = R_2 \mid R_2 = R_1 \]

is meant to exchange the registers \( R_1 \) and \( R_2 \) :)

There are at least two ways of implementing this exchange ...
Problem

The parallel register assignment:

\[ \psi_1 = R_1 = R_2 \mid R_2 = R_1 \]

is meant to exchange the registers \( R_1 \) and \( R_2 \) :-)

There are at least two ways of implementing this exchange ...

(1) Using an auxiliary register:

\[
R = R_1;
R_1 = R_2;
R_2 = R;
\]
(2) XOR:

\[
R_1 = R_1 \oplus R_2;
\]

\[
R_2 = R_1 \oplus R_2;
\]

\[
R_1 = R_1 \oplus R_2;
\]
(2) XOR:

\[ R_1 = R_1 \oplus R_2; \]
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But what about cyclic shifts such as:

\[ \psi_k = R_1 = R_2 | \ldots | R_{k-1} = R_k | R_k = R_1 \]

for \( k > 2 \)??
(2) **XOR:**

\[
\begin{align*}
R_1 &= R_1 \oplus R_2; \\
R_2 &= R_1 \oplus R_2; \\
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\]

But what about cyclic shifts such as:

\[
\psi_k = R_1 = R_2 | \ldots | R_{k-1} = R_k | R_k = R_1
\]

for \( k > 2 \) ??

Then at most \( k - 1 \) swaps of two registers are needed:

\[
\psi_k = R_1 \leftrightarrow R_2; \\
R_2 \leftrightarrow R_3; \\
\ldots \\
R_{k-1} \leftrightarrow R_k;
\]
Next complicated case: permutations.

- Every permutation can be decomposed into a set of disjoint shifts.
- Any permutation of $n$ registers with $r$ shifts can be realized by $n - r$ swaps...
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- Every permutation can be decomposed into a set of disjoint shifts
  :-)
- Any permutation of $n$ registers with $r$ shifts can be realized by $n - r$ swaps ...

Example

$$\psi = R_1 = R_2 \mid R_2 = R_5 \mid R_3 = R_4 \mid R_4 = R_3 \mid R_5 = R_1$$

consists of the cycles $(R_1, R_2, R_5)$ and $(R_3, R_4)$. Therefore:

$$\psi = R_1 \leftrightarrow R_2;$$
$$R_2 \leftrightarrow R_5;$$
$$R_3 \leftrightarrow R_4;$$
The general case:

- Every register receives its value at most once.
- The assignment therefore can be decomposed into a permutation together with tree-like assignments (directed towards the leaves) ...

Example

\[
\psi = R_1 = R_2 \mid R_2 = R_4 \mid R_3 = R_5 \mid R_5 = R_3
\]

The parallel assignment realizes the linear register moves for \(R_1, R_2\) and \(R_4\) together with the cyclic shift for \(R_3\) and \(R_5\):

\[
\psi = R_1 = R_2; \\
R_2 = R_4; \\
R_3 \leftrightarrow R_5;
\]