Generalization: Static Single Assignment Form

We proceed in two phases:

Step 1:

Transform the program such that each program point v is reached by at most one definition of a variable x which is live at v.

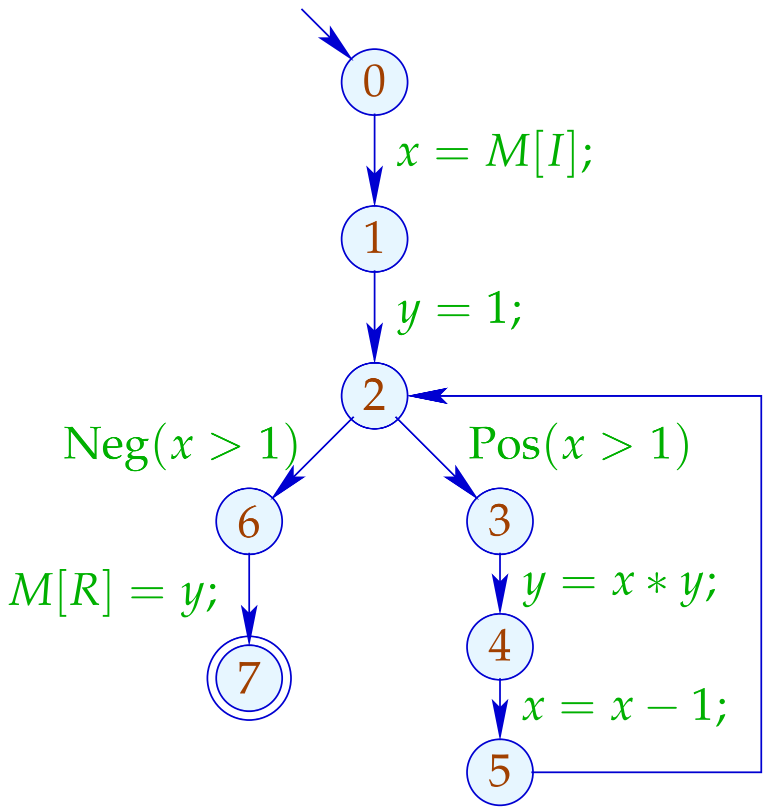
Step 2:

- Introduce a separate variant x_i for every occurrence of a definition of a variable x !
- Replace every use of x with the use of the reaching variant x_h ...

Implementing Step 1:

- Determine for every program point the set of reaching definitions.
- If the join point v is reached by more than one definition for the same variable x which is live at program point v, insert definitions $x = x;$ at the end of each incoming edge.

Example



Reaching Definitions

	$\mathcal{R}$
0	$\langle x, 0\rangle, \langle y, 0\rangle$
1	$\langle x, 1\rangle, \langle y, 0\rangle$
2	$\langle x, 1\rangle, \langle x, 5\rangle, \langle y, 2\rangle, \langle y, 4\rangle$
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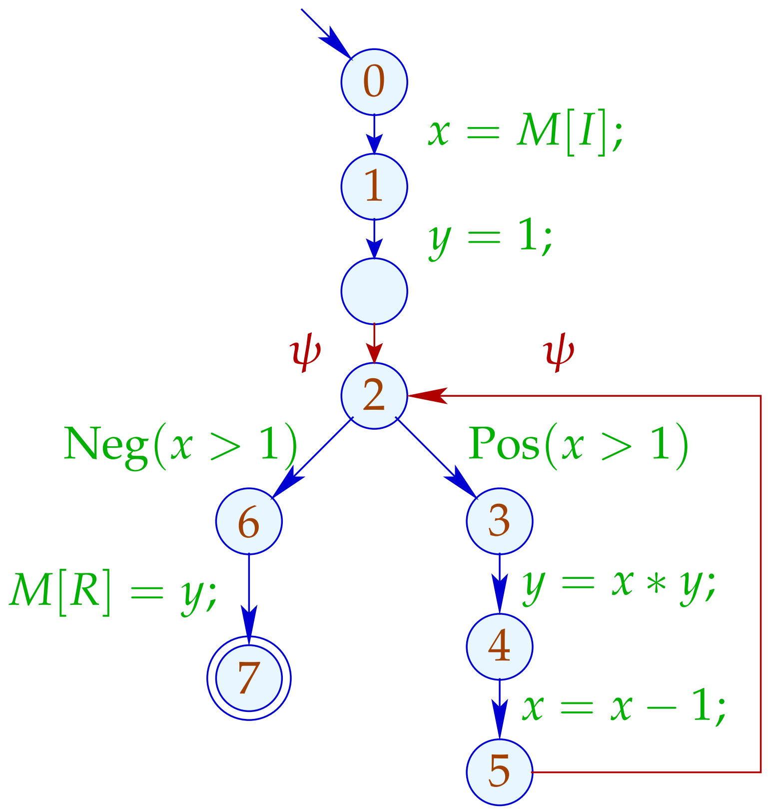

where $\psi \quad \equiv \quad x = x \mid y = y$

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Reaching Definitions

The complete lattice $\mathbb{R}$ for this analysis is given by:

$$\mathbb{R} = 2^{Defs}$$

where

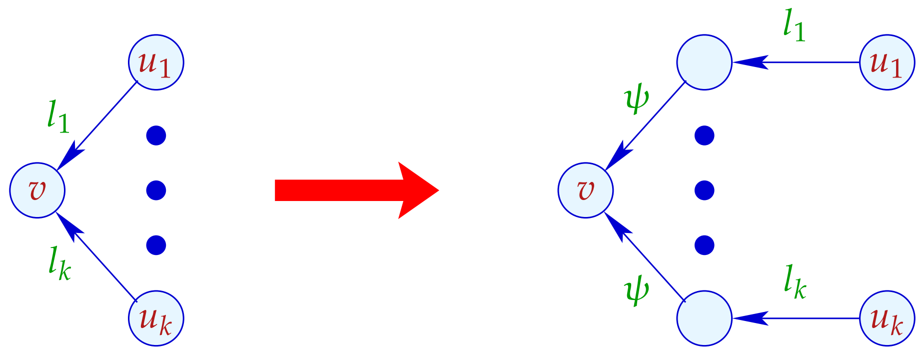
$$Defs = Vars \times Nodes \qquad Defs(x) = \{x\} \times Nodes$$

Then:

$$\begin{array}{lll}
[\![(_, x = r;, v)]\!]^\sharp R & = & R \backslash Defs(x) \cup \{\langle x, v \rangle\} \\
[\![(_, x = x \mid x \in L, v)]\!]^\sharp R & = & R \backslash \bigcup_{x \in L} Defs(x) \cup \{\langle x, v \rangle \mid x \in L\}
\end{array}$$

The ordering on $\mathbb{R}$ is given by subset inclusion $\subseteq$ where the value at program start is given by $R_0 = \{\langle x, start \rangle \mid x \in Vars\}$.

The Transformation SSA, Step 1:



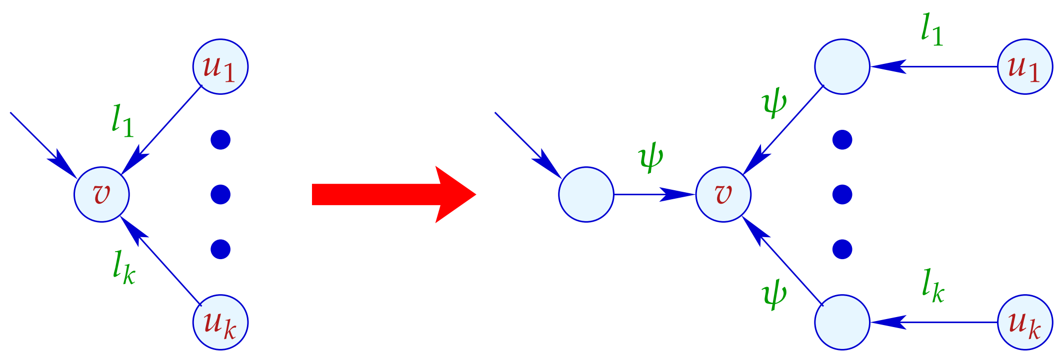
where $k \geq 2$.

The label ψ of the new in-going edges for v is given by:

$$\psi \;\equiv\; \{x = x \mid x \in \mathcal{L}[v], \#(\mathcal{R}[v] \cap \mathit{Defs}(x)) > 1\}$$

If the node v is the start point of the program, we add auxiliary edges whenever there are further ingoing edges into v:

The Transformation SSA, Step 1 (cont.):



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Discussion

- Program start is interpreted as (the end point of) a definition of every variable x :-)
- At some edges, parallel definitions ψ are introduced !
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Improvement:

- We introduce assignments $x = x$ before v only if the sets of reaching definitions for x at incoming edges of v differ !
- This introduction is repeated until every v is reached by exactly one definition for each variable live at v.

Theorem

Assume that every program point in the controlflow graph is reachable from start and that every left-hand side of a definition is live. Then:

1. The algorithm for inserting definitions $x = x$ terminates after at most $n \cdot (m + 1)$ rounds were m is the number of program points with more than one in-going edges and n is the number of variables.

2. After termination, for every program point u, the set $\mathcal{R}[u]$ has exactly one definition for every variable x which is live at u.

Discussion

The efficiency crucially depends on the number of iterations. If the cfg is well-structured, it terminates already after one iteration !

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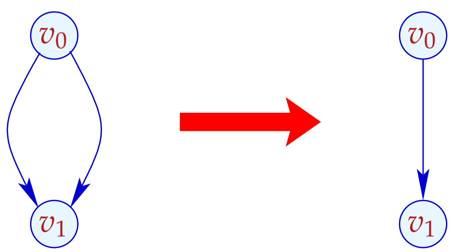
A well-structured cfg can be reduced to a single vertex or edge by:



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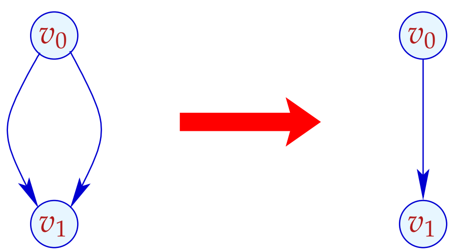
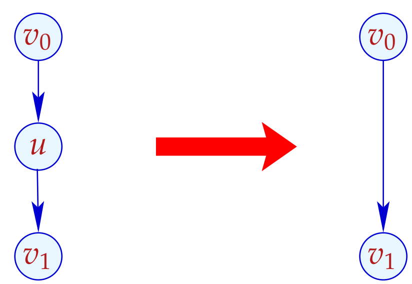




Discussion (cont.)

- Reducible cfgs are not the exception — but the rule :-)
- In Java, reducibility is only violated by loops with breaks/continues.
- If the insertion of definitions does not terminate after k iterations, we may immediately terminate the procedure by inserting definitions $x = x$ before all nodes which are reached by more than one definition of x.

Assume now that every program point u is reached by exactly one definition for each variable which is live at u ...

The Transformation SSA, Step 2:

Each edge (u, lab, v) is replaced with $(u, \mathcal{T}_{v,\phi}[lab], v)$ where $\phi\, x = x_{u'}$ if $\langle x, u' \rangle \in \mathcal{R}[u]$ and:

$$
\begin{array}{lll}
\mathcal{T}_{v,\phi}[\,;\,] & = & ; \\
\mathcal{T}_{v,\phi}[\mathsf{Neg}(e)] & = & \mathsf{Neg}(\phi(e)) \\
\mathcal{T}_{v,\phi}[\mathsf{Pos}(e)] & = & \mathsf{Pos}(\phi(e)) \\
\mathcal{T}_{v,\phi}[x = e] & = & x_v = \phi(e) \\
\mathcal{T}_{v,\phi}[x = M[e]] & = & x_v = M[\phi(e)] \\
\mathcal{T}_{v,\phi}[M[e_1] = e_2] & = & M[\phi(e_1)] = \phi(e_2)] \\
\mathcal{T}_{v,\phi}[\{x = x \mid x \in L\}] & = & \{x_v = \phi(x) \mid x \in L\}
\end{array}
$$

Remark

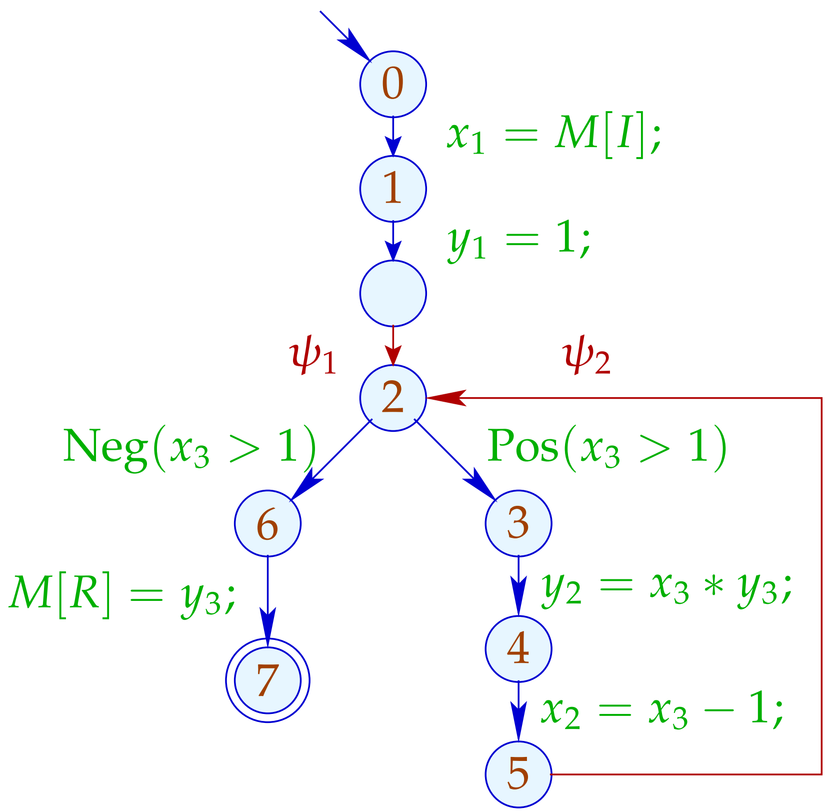
The multiple assignments:

$$pa = x_v^{(1)} = x_{v_1}^{(1)} \mid \ldots \mid x_v^{(k)} = x_{v_k}^{(k)}$$

in the last row are thought to be executed in parallel, i.e.,

$$[\![pa]\!]\,(\rho,\mu) = (\rho \oplus \{{x^{(i)}}_v \mapsto \rho({x^{(i)}}_{v_i}) \mid i = 1,\ldots,k\},\mu)$$

Example



$$\psi_1 \quad = \quad x_3 = x_1 \mid y_3 = y_1$$

$$\psi_2 \quad = \quad x_3 = x_2 \mid y_3 = y_2$$

Theorem

Assume that every program point is reachable from `start` and the program is in SSA form without assignments to dead variables.

Let λ denote the maximal number of simultaneously live variables and G the interference graph of the program variables. Then:

$$\lambda = \omega(G) = \chi(G)$$

where $\omega(G), \chi(G)$ are the maximal size of a clique in G and the minimal number of colors for G, respectively.

A minimal coloring of G, i.e., an optimal register allocation can be found in polynomial time.

Discussion

- By the theorem, the number λ of required registers can be easily computed :-)
- Thus variables which are to be spilled to memory, can be determined ahead of the subsequent assignment of registers !
- Thus here, we may, e.g., insist on keeping iteration variables from inner loops.

Discussion

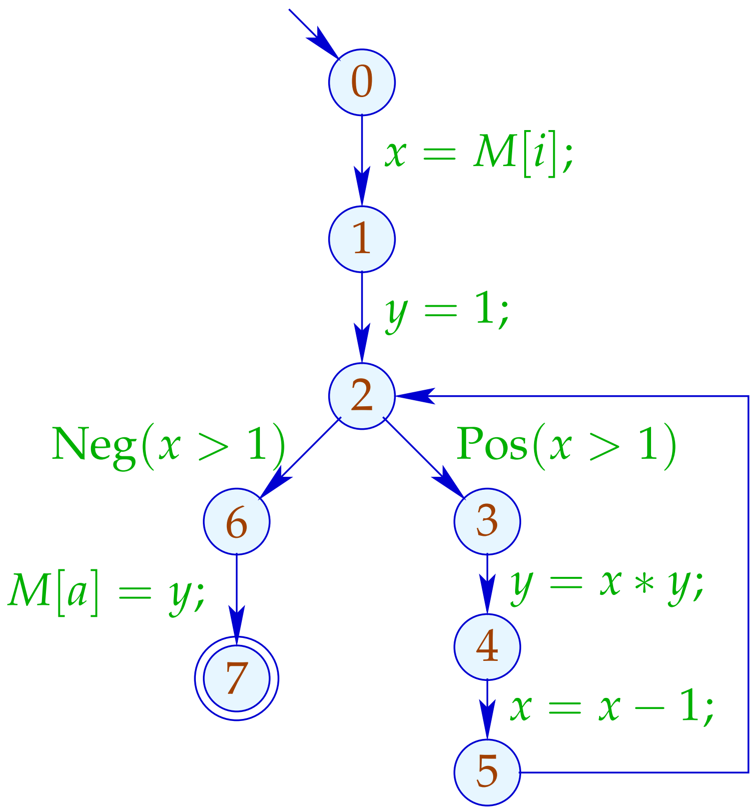
- By the theorem, the number λ of required registers can be easily computed :-)
- Thus variables which are to be spilled to memory, can be determined ahead of the subsequent assignment of registers !
- Thus here, we may, e.g., insist on keeping iteration variables from inner loops.
- Clearly, always $\lambda \leq \omega(G) \leq \chi(G)$:-)

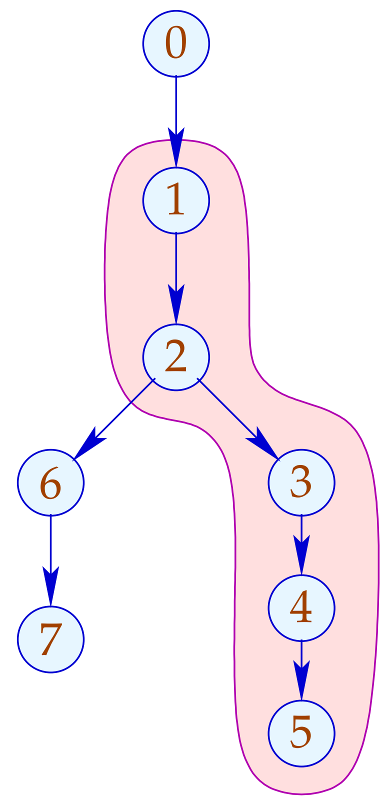
  Therefore, it suffices to color the interference graph with λ colors.
- Instead, we provide an algorithm which directly operates on the cfg ...

Observation

- Live ranges of variables in programs in SSA form behave similar to live ranges in basic blocks !
- Consider some dfs spanning tree T of the cfg with root start.
- For each variable x, the live range $\mathcal{L}[x]$ forms a tree fragment of T !
- A tree fragment is a subtree from which some subtrees have been removed ...

Example





Discussion

- Although the example program is not in SSA form, all live ranges still form tree fragments :-)
- The intersection of tree fragments is again a tree fragment !
- A set C of tree fragments forms a clique iff their intersection is non-empty !!!
- The greedy algorithm will find an optimal coloring ...

Proof of the Intersection Property

(1) Assume $I_1 \cap I_2 \neq \emptyset$ and v_i is the root of I_i. Then:

$$v_1 \in I_2 \quad \text{or} \quad v_2 \in I_1$$

(2) Let C denote a clique of tree fragments.
Then there is an enumeration $C = \{I_1, \ldots, I_r\}$ with roots $v_1, \ldots, v_r$ such that

$$v_i \in I_j \qquad \text{for all} \quad j \leq i$$

In particular, $v_r \in I_i$ for all i. :-)

The Greedy Algorithm

$$
\begin{array}{l}
\textbf{forall}\ (u \in \mathit{Nodes})\ \mathit{visited}[u] = \textbf{false};\\
\textbf{forall}\ (x \in \mathcal{L}[\mathit{start}])\ \Gamma(x) = \mathsf{extract}(\mathit{free});\\
\mathsf{alloc}(\mathit{start});\\
\\
\textbf{void}\ \mathsf{alloc}\ (\textbf{Node}\ u)\ \ \{\\
\quad \mathit{visited}[u] = \textbf{true};\\
\quad \textbf{forall}\ ((\mathit{lab}, v) \in \mathit{edges}[u])\\
\quad\quad \textbf{if}\ (\neg \mathit{visited}[v])\ \ \{\\
\quad\quad\quad \textbf{forall}\ (x \in \mathcal{L}[u] \backslash \mathcal{L}[v])\ \mathsf{insert}(\mathit{free}, x);\\
\quad\quad\quad \textbf{forall}\ (x \in \mathcal{L}[v] \backslash \mathcal{L}[u])\ \Gamma(x) = \mathsf{extract}(\mathit{free});\\
\quad\quad\quad \mathsf{alloc}\ (v);\\
\quad\quad\quad \}\\
\quad \}
\end{array}
$$

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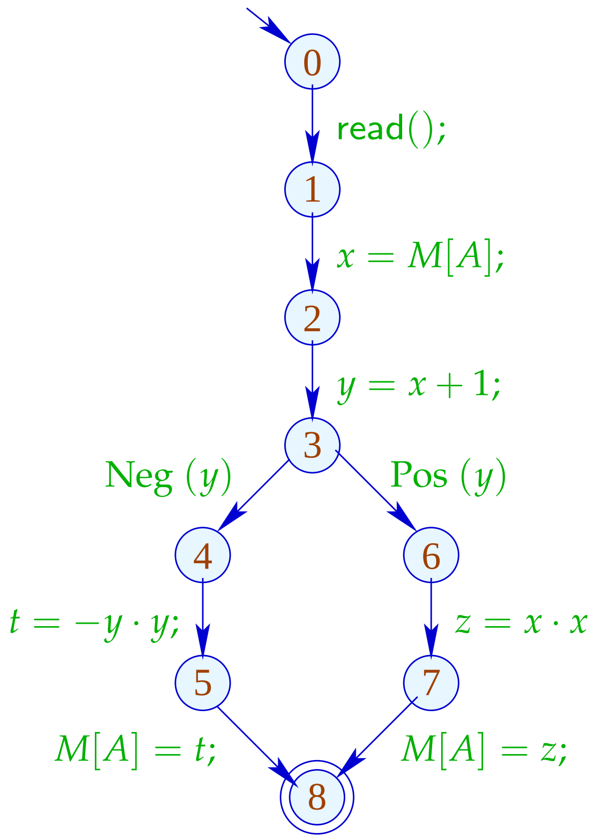

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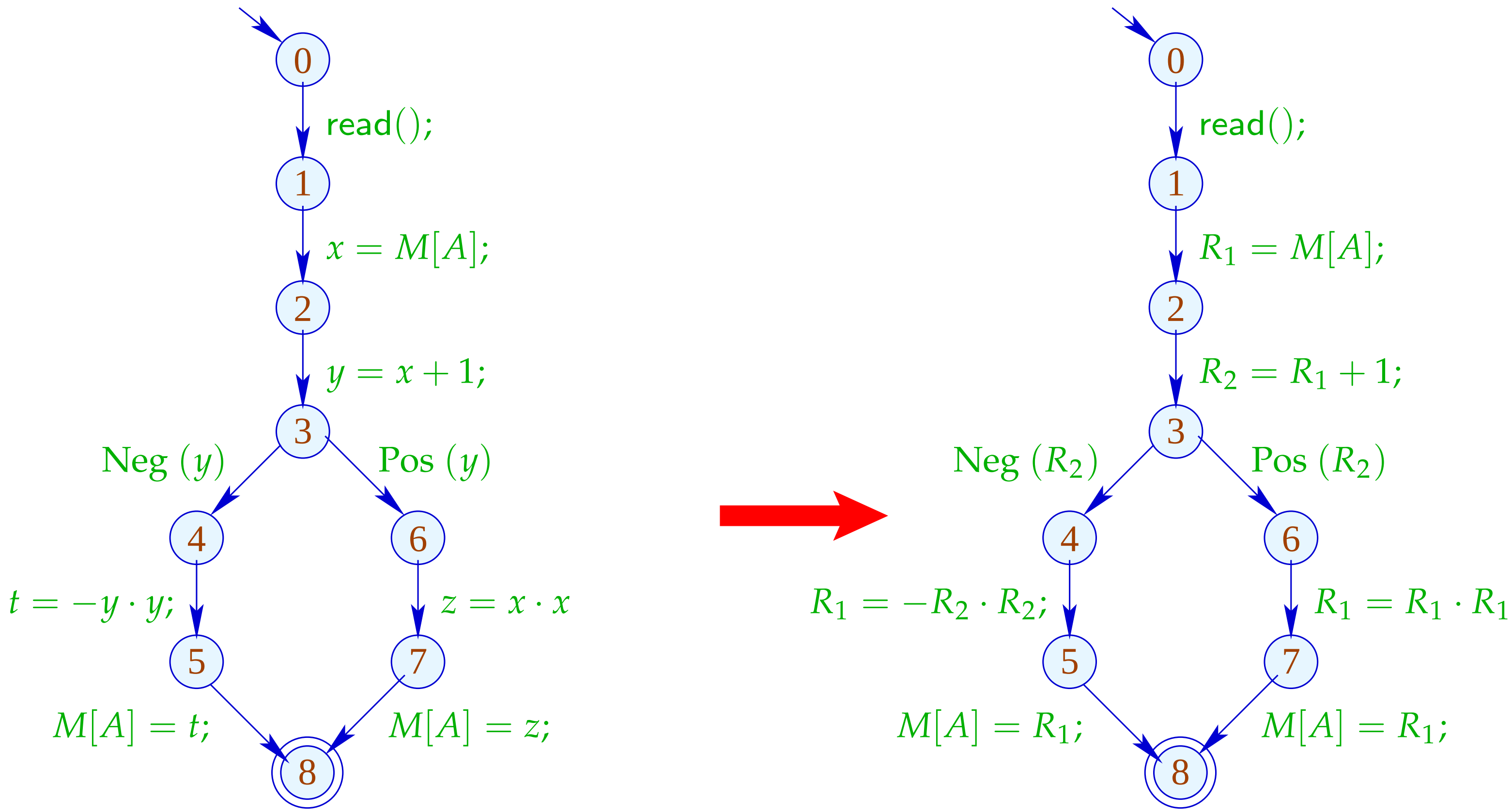

0
read();
1
x = M[A];
2
y = x + 1;
3
Neg (y)
Pos (y)
4
6
t = −y · y;
z = x · x
5
7
M[A] = t;
M[A] = z;
8
0
read();
1
R1 = M[A];
2
R2 = R1 + 1;
3
Neg (R2)
Pos (R2)
4
6
R1 = −R2 · R2;
R1 = R1 · R1
5
7
M[A] = R1;
M[A] = R1;
8

Remark:

- Intersection graphs for tree fragments are also known as cordal graphs ...
- A cordal graph is an undirected graph where every cycle with more than three nodes contains a cord :-)
- Cordal graphs are another sub-class of perfect graphs :-))

- Cheap register allocation comes at a price:

  when transforming into SSA form, we have introduced parallel register-register moves :-(

Problem

The parallel register assignment:

$$\psi_1 = R_1 = R_2 \mid R_2 = R_1$$

is meant to exchange the registers R_1 and R_2 :-)

There are at least two ways of implementing this exchange ...

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There are at least two ways of implementing this exchange ...

(1) Using an auxiliary register:

$$\begin{array}{lcl} R & = & R_1; \\ R_1 & = & R_2; \\ R_2 & = & R; \end{array}$$

(2) XOR:

$$\begin{aligned} R_1 &= R_1 \oplus R_2; \\ R_2 &= R_1 \oplus R_2; \\ R_1 &= R_1 \oplus R_2; \end{aligned}$$

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But what about cyclic shifts such as:

$$\psi_k = R_1 = R_2 \mid \ldots \mid R_{k-1} = R_k \mid R_k = R_1$$

for $k > 2$??

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Then at most $k-1$ swaps of two registers are needed:

$$\begin{aligned} \psi_k &= R_1 \leftrightarrow R_2; \\ & \quad R_2 \leftrightarrow R_3; \\ & \quad \ldots \\ & \quad R_{k-1} \leftrightarrow R_k; \end{aligned}$$

Next complicated case: permutations.

- Every permutation can be decomposed into a set of disjoint shifts :-)
- Any permutation of n registers with r shifts can be realized by $n - r$ swaps ...

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Example

$$\psi = R_1 = R_2 \mid R_2 = R_5 \mid R_3 = R_4 \mid R_4 = R_3 \mid R_5 = R_1$$

consists of the cycles (R_1, R_2, R_5) and (R_3, R_4). Therefore:

$$\begin{aligned} \psi \quad = \quad & R_1 \leftrightarrow R_2; \\ & R_2 \leftrightarrow R_5; \\ & R_3 \leftrightarrow R_4; \end{aligned}$$

The general case:

- Every register receives its value at most once.
- The assignment therefore can be decomposed into a permutation together with tree-like assignments (directed towards the leaves) ...

Example

$$\psi = R_1 = R_2 \mid R_2 = R_4 \mid R_3 = R_5 \mid R_5 = R_3$$

The parallel assignment realizes the linear register moves for R_1, R_2 and R_4 together with the cyclic shift for R_3 and R_5:

$$\begin{aligned} \psi \quad = \quad & R_1 = R_2; \\ & R_2 = R_4; \\ & R_3 \leftrightarrow R_5; \end{aligned}$$