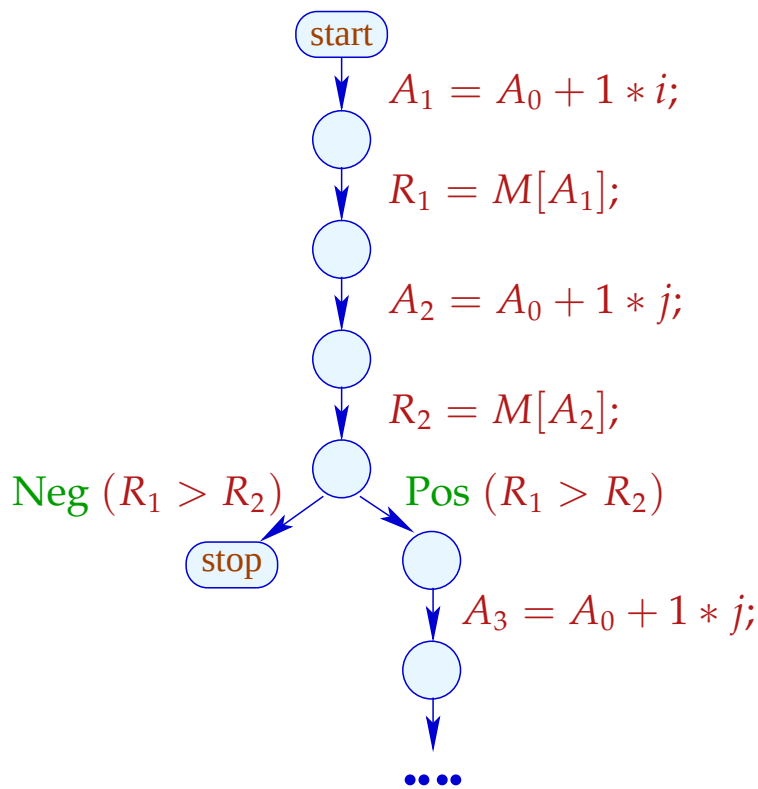
Background 1: An Operational Semantics

we choose a small-step operational approach.

Programs are represented as control-flow graphs.

In the example:



Thereby, represent:

vertex	program point
start	programm start
stop	program exit
edge	step of computation

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Edge Labelings:

Test : Pos (e) or Neg (e)

Assignment : $R = e;$

Load : $R = M[e];$

Store : $M[e_1] = e_2;$

Nop : $;$

Computations follow paths.

Computations transform the current state

$$s = (\rho, \mu)$$

where:

$\rho : Vars \to \mathbf{int}$	contents of registers
$\mu : \mathbb{N} \to \mathbf{int}$	contents of storage

Every edge $k = (u, lab, v)$ defines a partial transformation

$$[\![k]\!] = [\![lab]\!]$$

of the state:

$$
\begin{aligned}
[\![;]\!]\,(\rho,\mu) &= (\rho,\mu) \\
[\![\mathrm{Pos}\,(e)]\!]\,(\rho,\mu) &= (\rho,\mu) && \text{if } [\![e]\!]\,\rho \neq 0 \\
[\![\mathrm{Neg}\,(e)]\!]\,(\rho,\mu) &= (\rho,\mu) && \text{if } [\![e]\!]\,\rho = 0
\end{aligned}
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// $[\![e]\!]$: evaluation of the expression e, e.g.

// $[\![x+y]\!]\,\{x \mapsto 7, y \mapsto -1\} = 6$

// $[\![!(x == 4)]\!]\,\{x \mapsto 5\} = 1$

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$$[\![R = e;]\!]\,(\rho,\mu) \quad = \quad (\boxed{\rho \oplus \{R \mapsto [\![e]\!]\,\rho\}}, \mu)$$

// where "$\oplus$" modifies a mapping at a given argument

$$
\begin{aligned}
[\![R = M[e];]\!]\,(\rho, \mu) &= (\boxed{\rho \oplus \{R \mapsto \mu([\![e]\!]\,\rho))\}}, \mu) \\
[\![M[e_1] = e_2;]\!]\,(\rho, \mu) &= (\rho, \boxed{\mu \oplus \{[\![e_1]\!]\,\rho \mapsto [\![e_2]\!]\,\rho\}})
\end{aligned}
$$

Example:

$[\![x = x + 1;]\!]\,(\{x \mapsto 5\}, \mu) = (\rho, \mu)$ where:

$$
\begin{aligned}
\rho &= \{x \mapsto 5\} \oplus \{x \mapsto [\![x + 1]\!]\,\{x \mapsto 5\}\} \\
&= \{x \mapsto 5\} \oplus \{x \mapsto 6\} \\
&= \{x \mapsto 6\}
\end{aligned}
$$

A path $\pi = k_1 k_2 \ldots k_m$ is a computation for the state s if:

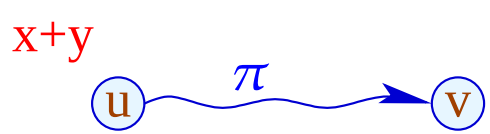
$$s \in def\,(\llbracket k_m \rrbracket \circ \ldots \circ \llbracket k_1 \rrbracket)$$

The result of the computation is:

$$\llbracket \pi \rrbracket\, s = (\llbracket k_m \rrbracket \circ \ldots \circ \llbracket k_1 \rrbracket)\, s$$

Application:

Assume that we have computed the value of $x + y$ at program point u:



We perform a computation along path π and reach v where we evaluate again $x + y$ ...

Idea:

If x and y have not been modified in π, then evaluation of $x + y$ at v must return the same value as evaluation at u :-)

We can check this property at every edge in π :-}

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More generally:

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More generally:

Assume that the values of the expressions $A = \{e_1, \ldots, e_r\}$ are available at u.

Every edge k transforms this set into a set $[\![k]\!]^\sharp\, A$ of expressions whose values are available after execution of k ...

... which transformations can be composed to the effect of a path $\pi = k_1 \ldots k_r$:

$$[\![\pi]\!]^\sharp = [\![k_r]\!]^\sharp \circ \ldots \circ [\![k_1]\!]^\sharp$$

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$$\begin{array}{lcl}
[\![;]\!]^\sharp\, A & = & A \\
[\![Pos(e)]\!]^\sharp\, A & = & [\![Neg(e)]\!]^\sharp\, A \quad = \quad A \cup \{e\} \\
[\![x = e;]\!]^\sharp\, A & = & (A \cup \{e\}) \backslash Expr_x \qquad \text{where} \\
 & & Expr_x \text{ all expressions which contain } x
\end{array}$$

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\begin{aligned}
[\![x = M[e];]\!]^\sharp\, A &= (A \cup \{e\}) \backslash Expr_x \\
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By that, every path can be analyzed :-)

A given program may admit several paths :-(

For any given input, another path may be chosen :-((

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$\Longrightarrow$ We require the set:

$$
\mathcal{A}[v] \quad = \quad \bigcap \{ [\![\pi]\!]^\sharp \emptyset \mid \pi : start \rightarrow^* v \}
$$

Concretely:

- $\longrightarrow$ We consider all paths π which reach v.
- $\longrightarrow$ For every path π, we determine the set of expressions which are available along π.
- $\longrightarrow$ Initially at program start, nothing is available :-)
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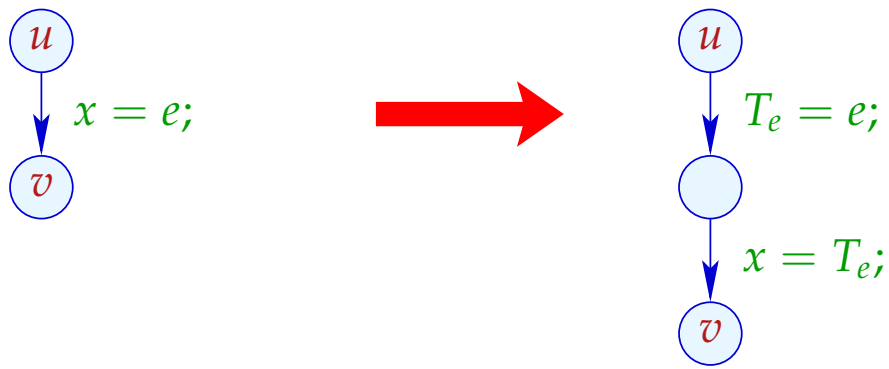
How do we exploit this information ???

Transformation 1.1:

We provide novel registers T_e as storage for the e:



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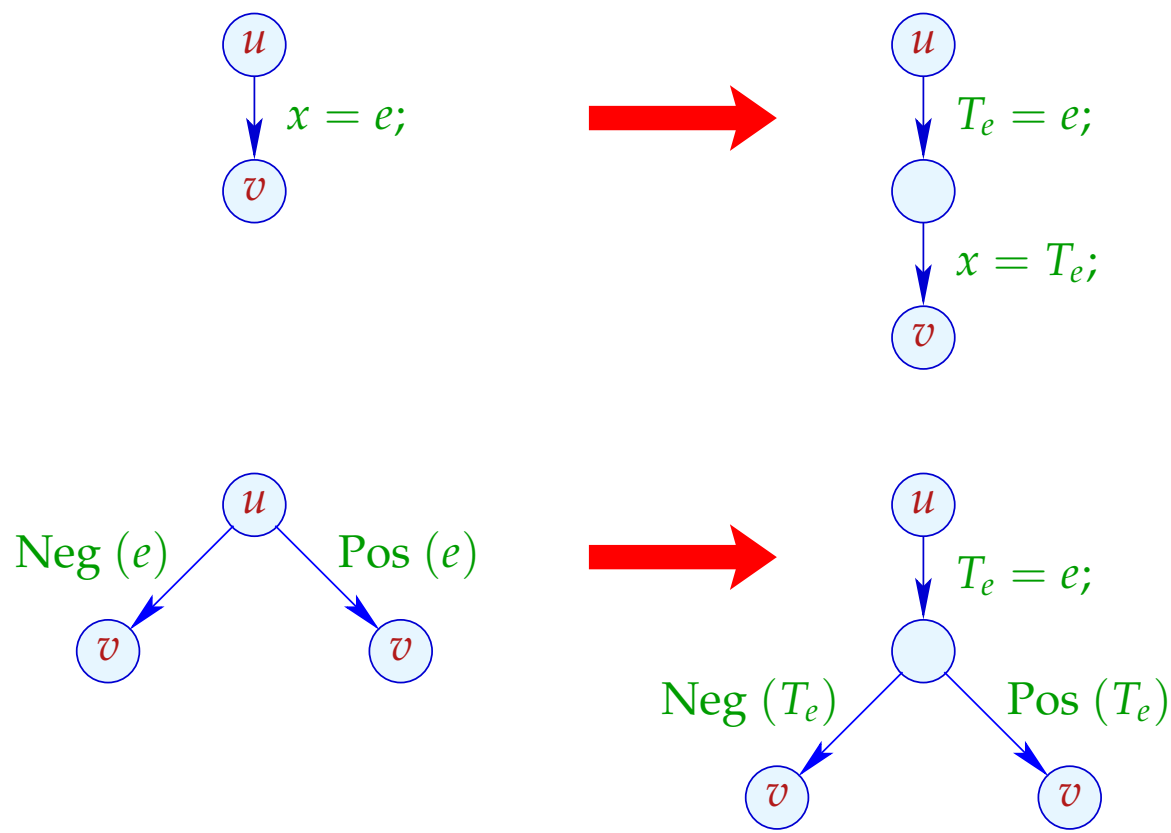


... analogously for $R = M[e];$ and $M[e_1] = e_2;$.

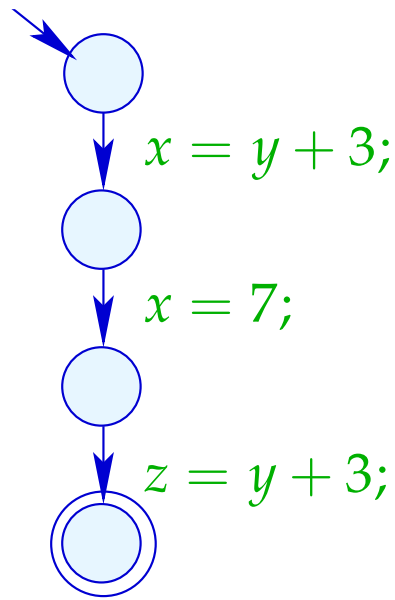
Transformation 1.2:

If e is available at program point u, then e need not be re-evaluated:

We replace the assignment with Nop :-)

Example:

$$
\begin{aligned}
x &= y + 3; \\
x &= 7; \\
z &= y + 3;
\end{aligned}
$$



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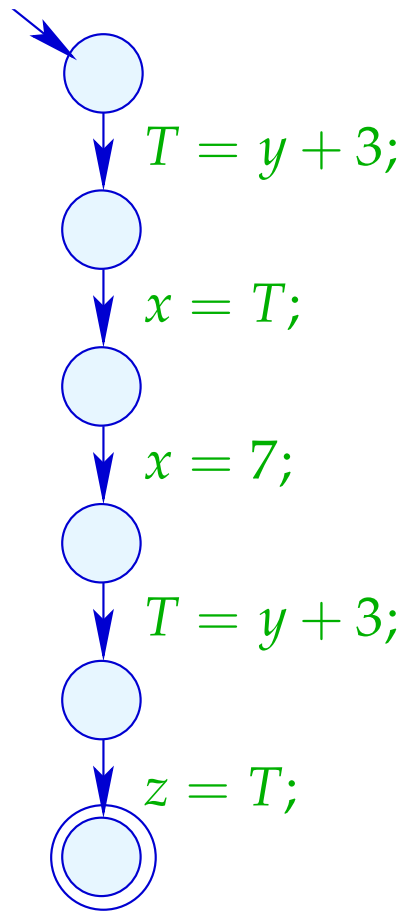

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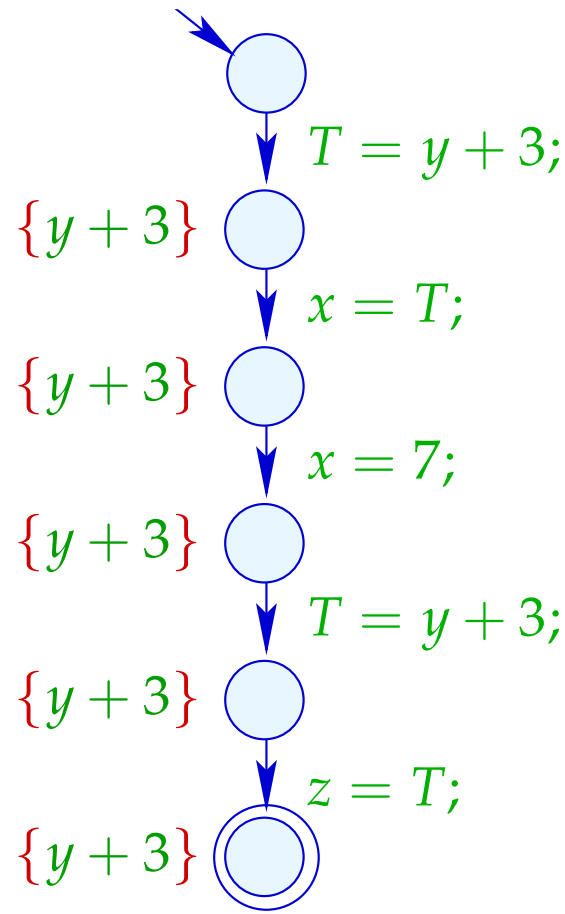

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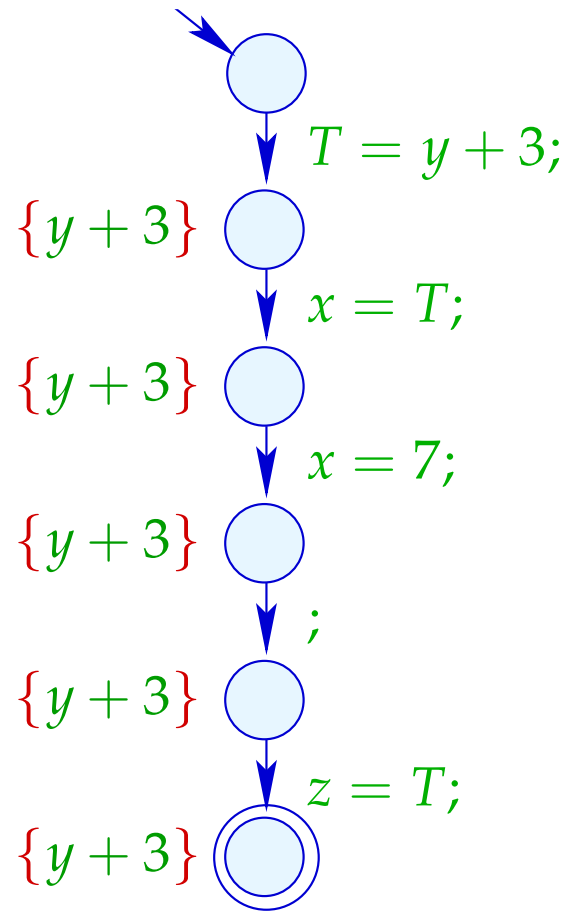


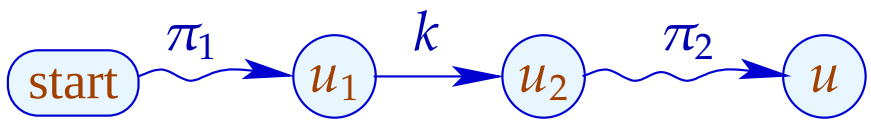
Correctness: (Idea)

Transformation 1.1 preserves the semantics and $\mathcal{A}[u]$ for all program points u :-)

Assume $\pi : start \rightarrow^* u$ is the path taken by a computation.

If $e \in \mathcal{A}[u]$, then also $e \in [\![\pi]\!]^\sharp\, \emptyset$.

Therefore, π can be decomposed into:



with the following properties:

- The expression e is evaluated at the edge k;
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$\Longrightarrow$

The register T_e contains the value of e whenever u is reached :-))

Warning:

Transformation 1.1 is only meaningful for assignments $x = e;$ where:

- $\rightarrow$ $x \notin Vars(e);$
- $\rightarrow$ $e \notin Vars;$
- $\rightarrow$ the evaluation of e is non-trivial :-}

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Which leaves us with the following question ...

Question:

How do we compute $\mathcal{A}[u]$ for every program point u ??