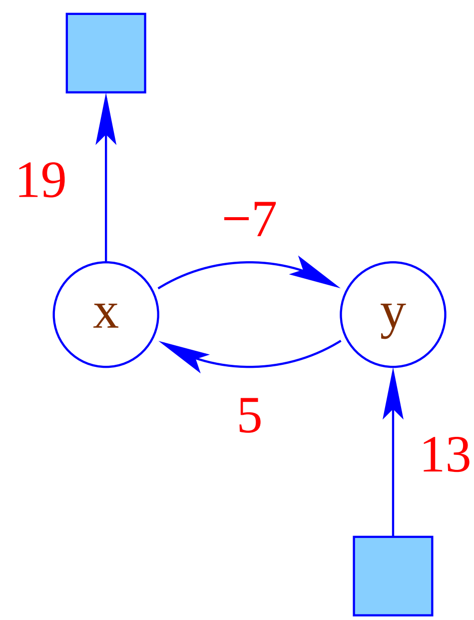
2. One Polynomial Special Case:

$$\begin{array}{rcr} x & \geq & y + 5 \\ 19 \geq & x & \\ y & \geq & 13 \\ y & \geq & x - 7 \end{array}$$

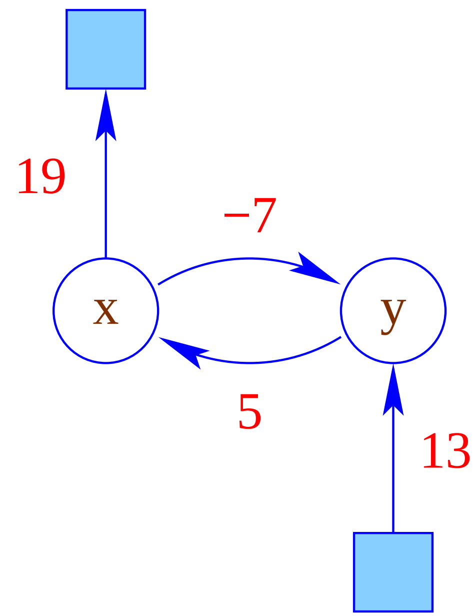
- There are at most 2 variables per in-equation;
- no scaling factors.

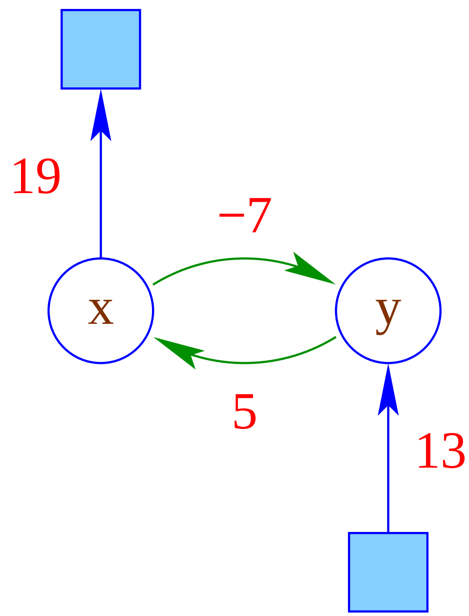
Idea: Represent the system by a graph:

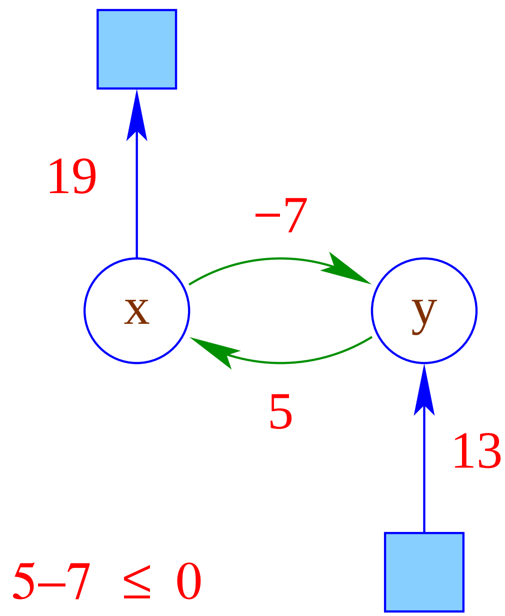


The in-equations are satisfiable iff

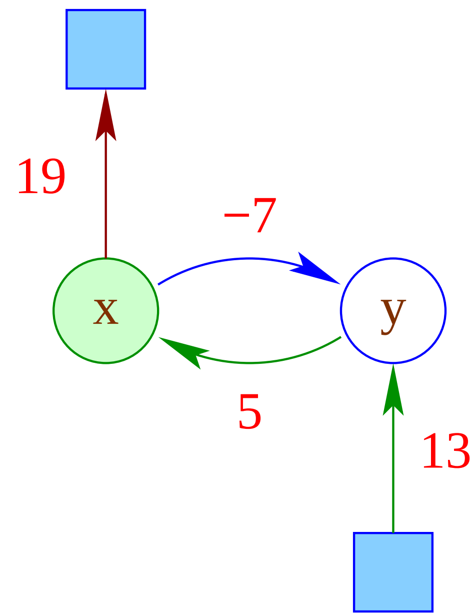
- the weight of every cycle are at most 0;
- the weights of paths reaching x are bounded by the weights of edges from x into the sink.

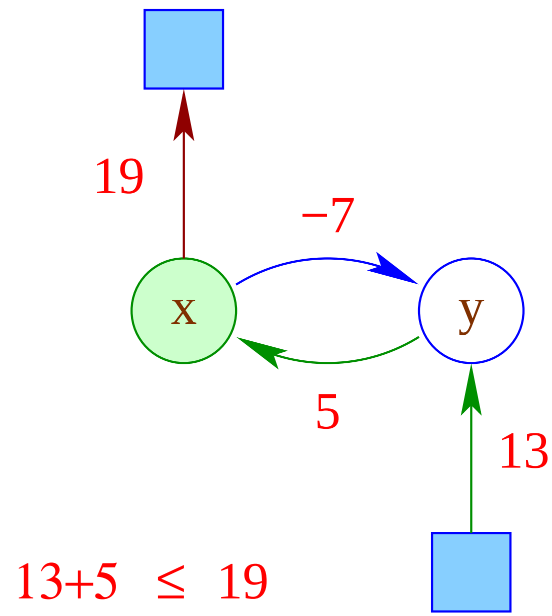

19
−7
x
y
5
13


19
−7
x
y
5
13



19
−7
x
y
5
13
5−7 ≤ 0


19
−7
x
y
5
13


19
−7
x
y
5
13
13+5 ≤ 19

The in-equations are satisfiable iff

- the weight of every cycle are at most 0;
- the weights of paths reaching x are bounded by the weights of edges from x into the sink.

$$\Longrightarrow$$

Compute the reflexive and transitive closure of the edge weights!

3. A General Solution Method:

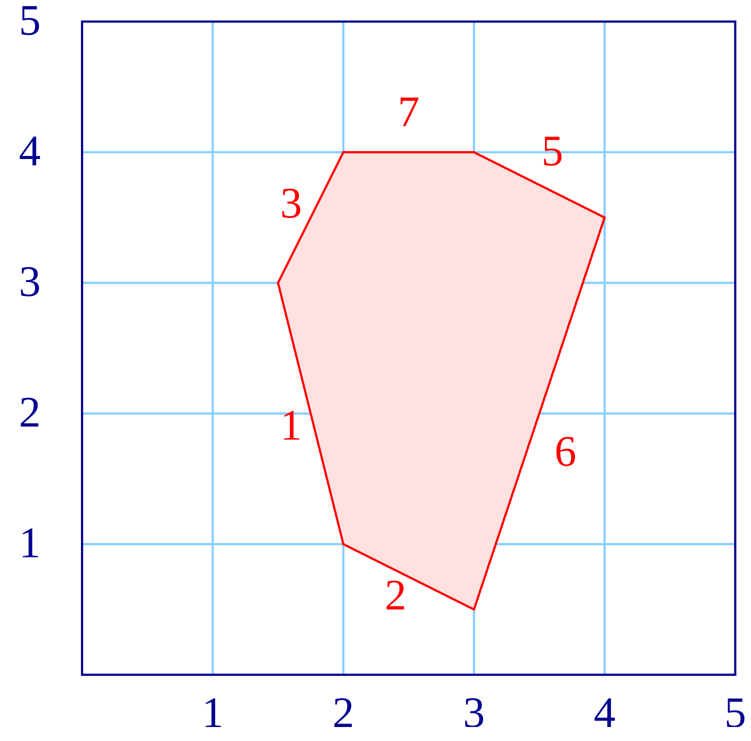
Idea: Fourier-Motzkin Elimination

- Successively remove individual variables x !
- All in-equations with positive occurrences of x yield lower bounds.
- All in-equations with negative occurrences of x yield upper bounds.
- All lower bounds must be at most as big as all upper bounds ;-))

Jean Baptiste Joseph Fourier, 1768–1830

Example:

$$\begin{array}{rcll} 9 & \leq & 4x_1 + x_2 & (1) \\ 4 & \leq & x_1 + 2x_2 & (2) \\ 0 & \leq & 2x_1 - x_2 & (3) \\ 6 & \leq & x_1 + 6x_2 & (4) \\ -11 & \leq & -x_1 - 2x_2 & (5) \\ -17 & \leq & -6x_1 + 2x_2 & (6) \\ -4 & \leq & -x_2 & (7) \end{array}$$



For x_1 we obtain:

$$\begin{array}{rcll} 9 & \leq & 4x_1 + x_2 & (1) \\ 4 & \leq & x_1 + 2x_2 & (2) \\ 0 & \leq & 2x_1 - x_2 & (3) \\ 6 & \leq & x_1 + 6x_2 & (4) \\ -11 & \leq & -x_1 - 2x_2 & (5) \\ -17 & \leq & -6x_1 + 2x_2 & (6) \\ -4 & \leq & -x_2 & (7) \end{array}$$

$$\begin{array}{rcll} \frac{9}{4} - \frac{1}{4}x_2 & \leq & x_1 & (1) \\ 4 - 2x_2 & \leq & x_1 & (2) \\ \frac{1}{2}x_2 & \leq & x_1 & (3) \\ 6 - 6x_2 & \leq & x_1 & (4) \\ x_1 & \leq & 11 - 2x_2 & (5) \\ x_1 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (6) \\ -4 & \leq & -x_2 & (7) \end{array}$$

If such an x_1 exists, all lower bounds must be bounded by all upper bounds, i.e.,

$$
\begin{array}{rclcrclc}
\frac{9}{4} - \frac{1}{4}x_2 & \leq & 11 - 2x_2 & (1,5) & & -35 & \leq & -7x_2 & (1,5) \\
\frac{9}{4} - \frac{1}{4}x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (1,6) & & -\frac{7}{12} & \leq & \frac{7}{12}x_2 & (1,6) \\
4 - 2x_2 & \leq & 11 - 2x_2 & (2,5) & & -7 & \leq & 0 & (2,5) \\
4 - 2x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (2,6) & & \frac{7}{6} & \leq & \frac{7}{3}x_2 & (2,6) \\
\frac{1}{2}x_2 & \leq & 11 - 2x_2 & (3,5) & \text{or} & -22 & \leq & -5x_2 & (3,5) \\
\frac{1}{2}x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (3,6) & & -\frac{17}{6} & \leq & -\frac{1}{6}x_2 & (3,6) \\
6 - 6x_2 & \leq & 11 - 2x_2 & (4,5) & & -5 & \leq & 4x_2 & (4,5) \\
6 - 6x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (4,6) & & \frac{19}{6} & \leq & \frac{19}{3}x_2 & (4,6) \\
-4 & \leq & -x_2 & (7) & & -4 & \leq & -x_2 & (7)
\end{array}
$$

$$
\begin{array}{rclcrclc}
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\frac{9}{4} - \frac{1}{4}x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (1,6) & & -1 & \leq & x_2 & (1,6) \\
4 - 2x_2 & \leq & 11 - 2x_2 & (2,5) & & -7 & \leq & 0 & (2,5) \\
4 - 2x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (2,6) & & \frac{1}{2} & \leq & x_2 & (2,6) \\
\frac{1}{2}x_2 & \leq & 11 - 2x_2 & (3,5) & \text{or} & -\frac{22}{5} & \leq & -x_2 & (3,5) \\
\frac{1}{2}x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (3,6) & & -17 & \leq & -x_2 & (3,6) \\
6 - 6x_2 & \leq & 11 - 2x_2 & (4,5) & & -\frac{5}{4} & \leq & x_2 & (4,5) \\
6 - 6x_2 & \leq & \frac{17}{6} + \frac{1}{3}x_2 & (4,6) & & \frac{1}{2} & \leq & x_2 & (4,6) \\
-4 & \leq & -x_2 & (7) & & -4 & \leq & -x_2 & (7)
\end{array}
$$

This is the one-variable case which we can solve exactly:

$$\max \{-1, \boxed{\tfrac{1}{2}}, -\tfrac{5}{4}, \tfrac{1}{2}\} \quad \leq \quad x_2 \quad \leq \quad \min \{5, \tfrac{22}{5}, 17, \boxed{4}\}$$

From which we conclude: $x_2 \in [\frac{1}{2}, 4]$:-)

In General:

- The original system has a solution over $\mathbb{Q}$ iff the system after elimination of one variable has a solution over $\mathbb{Q}$:-)
- Every elimination step may square the number of in-equations $\Longrightarrow$ exponential run-time :-((
- It can be modified such that it also decides satisfiability over $\mathbb{Z}$ $\Longrightarrow$ Omega Test

William Worthington Pugh, Jr.
University of Maryland, College Park

Idea:

- We successively remove variables. Thereby we omit division ...
- If x only occurs with coefficient ± 1, we apply Fourier-Motzkin elimination :-)
- Otherwise, we provide a bound for a positive multiple of x ...

Consider, e.g., (1) and (6) :

$$\begin{array}{rcl} 6 \cdot x_1 & \leq & 17 + 2x_2 \\ 9 - x_2 & \leq & 4 \cdot x_1 \end{array}$$

W.l.o.g., we only consider strict in-equations:

$$\begin{aligned} 6 \cdot x_1 &< 18 + 2x_2 \\ 8 - x_2 &< 4 \cdot x_1 \end{aligned}$$

... where we always divide by gcds:

$$\begin{aligned} 3 \cdot x_1 &< 9 + x_2 \\ 8 - x_2 &< 4 \cdot x_1 \end{aligned}$$

This implies:

$$3 \cdot (8 - x_2) < 4 \cdot (9 + x_2)$$

We thereby obtain:

- If one derived in-equation is unsatisfiable, then also the overall system :-)
- If all derived in-equations are satisfiable, then there is a solution which, however, need not be integer :-(
- An integer solution is guaranteed to exist if there is sufficient separation between lower and upper bound ...
- Assume $\alpha < a \cdot x \qquad b \cdot x < \beta$.

  Then it should hold that:

  $$b \cdot \alpha < a \cdot \beta$$

  and moreover:

  $$\boxed{a \cdot b} < a \cdot \beta - b \cdot \alpha$$

... in the Example:

$$12 \quad < \quad 4 \cdot (9 + x_2) - 3 \cdot (8 - x_2)$$

or:

$$12 \quad < \quad 12 + 7x_2$$

or:

$$0 \quad < \quad x_2$$

In the example, also these strengthened in-equations are satisfiable

$\Longrightarrow$ the system has a solution over $\mathbb{Z}$:-)

Discussion:

- If the strengthened in-equations are satisfiable, then also the original system. The reverse implication may be wrong :-(
- In the case where upper and lower bound are not sufficiently separated, we have:

$$a \cdot \beta \leq b \cdot \alpha + \boxed{a \cdot b}$$

or:

$$b \cdot \alpha < ab \cdot x < b \cdot \alpha + \boxed{a \cdot b}$$

Division with b yields:

$$\alpha < a \cdot x < \alpha + \boxed{a}$$

$\Longrightarrow$ $\boxed{\alpha + i = a \cdot x}$ for some $i \in \{1, \ldots, a - 1\}$!!!

Discussion (cont.):

- → Fourier-Motzkin Elimination is not the best method for rational systems of in-equations.
- → The Omega test is necessarily exponential :-)

  If the system is solvable, the test generally terminates rapidly.

  It may have problems with unsolvable systems :-(
- → Also for ILP, there are other/smarter algorithms ...
- → For programming language problems, however, it seems to behave quite well :-)

4. Generalization to a Logic

Disjunction:

$$\begin{array}{llll} (x - 2y = 15 & \wedge & x + y = 7) & \vee \\ (x + y = 6 & \wedge & 3x + z = -8) & \end{array}$$

Quantors:

$$\exists\, x : \quad z - 2x = 42 \quad \wedge \quad z + x = 19$$

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$\Longrightarrow$ Presburger Arithmetic

Mojzesz Presburger, 1904–1943 (?)

Presburger Arithmetic $=$ full arithmetic without multiplication

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Arithmetic : highly undecidable :-(

even incomplete :-((

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$\Longrightarrow$ Hilbert's 10th Problem

$\Longrightarrow$ Gödel's Theorem

Presburger Formulas over $\mathbb{N}$:

$$\begin{array}{lll} \phi & ::= & x + y = z \quad | \quad x = n \quad | \\ & & \phi_1 \wedge \phi_2 \quad | \quad \neg\phi \quad | \\ & & \exists\, x : \quad \phi \end{array}$$

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Goal: PSAT

Find values for the free variables in $\mathbb{N}$ such that ϕ holds ...

Idea: Code the values of the variables as Words :-)

213	t	1	0	1	0	1	0	1	1
42	z	0	1	0	1	0	1	0	0
89	y	1	0	0	1	1	0	1	0
17	x	1	0	0	0	1	0	0	0

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Observation:

The set of satisfying variable assignments is regular :-))

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$$
\begin{array}{llll}
\phi_1 \wedge \phi_2 & \Longrightarrow & \mathcal{L}(\phi_1) \cap \mathcal{L}(\phi_2) & \text{(Intersection)} \\
\neg \phi & \Longrightarrow & \overline{\mathcal{L}(\phi)} & \text{(Complement)} \\
\exists\, x : \quad \phi & \Longrightarrow & \pi_x(\mathcal{L}(\phi)) & \text{(Projection)}
\end{array}
$$

Projecting away the x-component:

213	t	1	0	1	0	1	0	1	1
42	z	0	1	0	1	0	1	0	0
89	y	1	0	0	1	1	0	1	0
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Projecting away the x-component:

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42	z	0	1	0	1	0	1	0	0
89	y	1	0	0	1	1	0	1	0

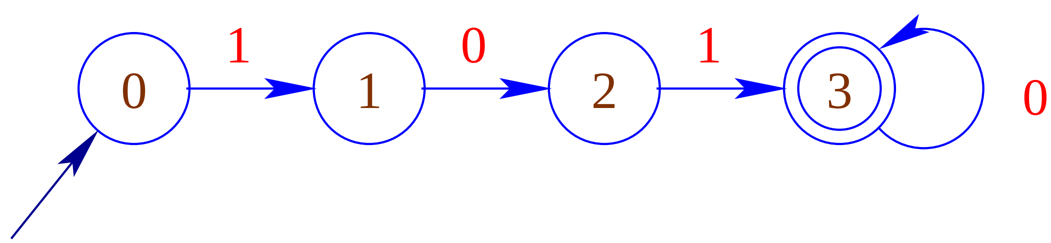
Warning:

- Our representation of numbers is not unique: 011101 should be accepted iff every word from $011101 \cdot 0^*$ is accepted!
- This property is preserved by union, intersection and complement :-)
- It is lost by projection !!!

$\Longrightarrow$ The automaton for projection must be enriched such that the property is re-established !!

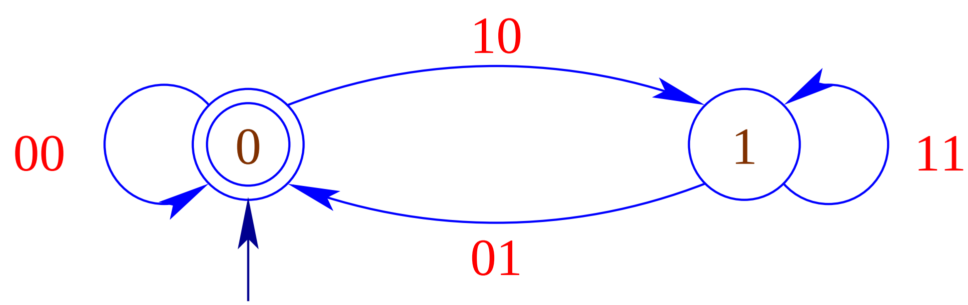
Automata for Basic Predicates:

$x = 5$



Automata for Basic Predicates:

$x+x = y$



Automata for Basic Predicates:

$x+y = z$