4 Optimization of Functional Programs

Example:

$$\begin{aligned} \textbf{let rec } \mathsf{fac}\ x \quad = \quad & \textbf{if } x \leq 1 \textbf{ then } 1 \\ & \textbf{else } x \cdot \mathsf{fac}\ (x-1) \end{aligned}$$

- There are no basic blocks :-(
- There are no loops :-(
- Virtually all functions are recursive :-((

Strategies for Optimization:

⟹ Improve specific inefficiencies such as:

- Pattern matching
- Lazy evaluation (if supported ;-)
- Indirections — Unboxing / Escape Analysis
- Intermediate data-structures — Deforestation

⟹ Detect and/or generate loops with basic blocks :-)

- Tail recursion
- Inlining
- **let**-Floating

Then apply general optimization techniques

... e.g., by translation into C ;-)

Warning:

Novel analysis techniques are needed to collect information about functional programs.

Example: Inlining

$$\begin{aligned}\textbf{let}\ \ \mathsf{max}\ (x, y) &= \textbf{if}\ \ x > y\ \ \textbf{then}\ \ x \\ &\quad\ \textbf{else}\ \ y \\ \textbf{let}\ \ \mathsf{abs}\ z &= \mathsf{max}\ (z, -z)\end{aligned}$$

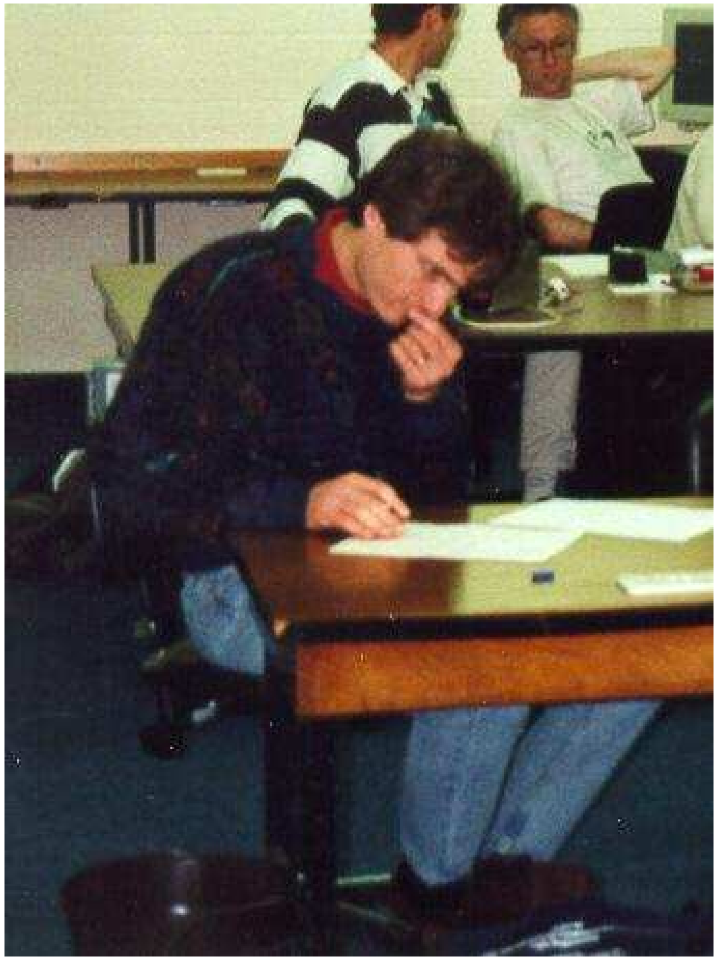
As result of the optimization we expect ...

$$
\begin{array}{llll}
\textbf{let}\ \mathsf{max}\ (x, y) & = & \textbf{if}\ \ x > y\ \ \textbf{then}\ \ x & \\
 & & \textbf{else}\ \ y & \\
\textbf{let}\ \mathsf{abs}\ z & = & \textbf{let} & x = z \\
 & & \textbf{and} & y = -z \\
 & & \textbf{in} & \boxed{\begin{array}{l}\textbf{if}\ \ x > y\ \ \textbf{then}\ \ x \\ \textbf{else}\ \ y\end{array}} \\
 & & \textbf{end} &
\end{array}
$$

Discussion:

For the beginning, max is just a name. We must find out which value it takes at run-time

$\Longrightarrow$ Value Analysis required !!



Nevin Heintze in the Australian team
of the Prolog-Programming-Contest, 1998

The complete picture:

4.1 A Simple Functional Language

For simplicity, we consider:

$$
\begin{array}{lcl}
e & ::= & b \mid (e_1, \ldots, e_k) \mid c\ e_1\ \ldots\ e_k \mid \textbf{fun}\, x \rightarrow e \\
 & & \mid (e_1\, e_2) \mid (\Box_1\ e) \mid (e_1\, \Box_2\, e_2) \mid \\
 & & \textbf{let}\ x_1 = e_1\ \textbf{and} \ldots \textbf{and}\ x_k = e_k\ \textbf{in}\ e_0 \mid \\
 & & \textbf{match}\ e_0\ \textbf{with}\ p_1 \rightarrow e_1\ \mid \ldots \mid\ p_k \rightarrow e_k \\
p & ::= & b \mid x \mid c\, x_1 \ldots x_k \mid (x_1, \ldots, x_k) \\
t & ::= & \textbf{let rec}\ x_1 = e_1\ \textbf{and} \ldots \textbf{and}\ x_k = e_k\ \textbf{in}\ e
\end{array}
$$

where b is a constant, x is a variable, c is a (data-)constructor and $\Box_i$ are i-ary operators.

Discussion:

- **let rec** only occurs on top-level.
- Functions are always unary. Instead, there are explicit tuples :-)
- **if**-expressions and case distinction in function definitions is reduced to **match**-expressions.
- In case distinctions, we allow just simple patterns.

  $\Longrightarrow$ Complex patterns must be decomposed ...
- **let**-definitions correspond to basic blocks :-)
- Type-annotations at variables, patterns or expressions could provide further useful information

  — which we ignore :-)

... in the Example:

A definition of max may look as follows:

$$
\begin{array}{l}
\textbf{let } \mathsf{max} \;=\; \textbf{fun } x \rightarrow \;\; \textbf{match } x \textbf{ with } (x_1, x_2) \;\rightarrow\; (\\
\qquad\qquad\qquad\qquad \textbf{match } x_1 < x_2 \\
\qquad\qquad\qquad\qquad \;\textbf{with } \;\mathsf{True} \;\rightarrow\; x_2 \\
\qquad\qquad\qquad\qquad \quad\;\; \mid \;\;\; \mathsf{False} \;\rightarrow\; x_1 \\
\qquad\qquad\qquad\qquad)
\end{array}
$$

Accordingly, we have for abs :

$$\textbf{let}\ \mathsf{abs}\ =\ \textbf{fun}\ x \rightarrow\ \ \textbf{let}\ z = (x, -x)$$
$$\textbf{in}\ \mathsf{max}\ z$$

4.2 A Simple Value Analysis

Idea:

For every subexpression e we collect the set $[\![e]\!]^\sharp$ of possible values of e ...

Let V denote the set of occurring (classes of) constants, functions as well as applications of constructors and operators. As our lattice, we choose:

$$\mathbb{V} = 2^V$$

As usual, we put up a constraint system:

- If e is a value, i.e., of the form: $b, c\, e_1 \ldots e_k, (e_1, \ldots, e_k)$, an operator application or $\textbf{fun}\ x \to e$ we generate the constraint:

$$[\![e]\!]^\sharp \supseteq \{e\}$$

- If $e \equiv (e_1\ e_2)$ and $f \equiv \textbf{fun}\ x \to e'$, then

$$[\![e]\!]^\sharp \supseteq (f \in [\![e_1]\!]^\sharp)\ ?\ [\![e']\!]^\sharp : \emptyset$$
$$[\![x]\!]^\sharp \supseteq (f \in [\![e_1]\!]^\sharp)\ ?\ [\![e_2]\!]^\sharp : \emptyset$$

...

- int-values returned by operators are described by the unevaluated expression;

  Operator applications which return Boolean values, e.g., by $\{\mathsf{True}, \mathsf{False}\}$:-)

- If $e \equiv \textbf{let } x_1 = e_1 \textbf{ and} \ldots \textbf{and } x_k = e_k \textbf{ in } e_0$, then we generate:

$$\begin{array}{lll} [\![x_i]\!]^\sharp & \supseteq & [\![e_i]\!]^\sharp \\ [\![e]\!]^\sharp & \supseteq & [\![e_0]\!]^\sharp \end{array}$$

- Assume $e \equiv \textbf{match}\ e_0\ \textbf{with}\ p_1 \ \rightarrow\ e_1 \mid \ldots \mid p_k \ \rightarrow\ e_k$.
  Then we generate for $p_i \equiv b$,

$$[\![e]\!]^\sharp \supseteq [\![e_i]\!]^\sharp \ : \ \emptyset$$

  If $p_i \equiv c\, y_1 \ldots y_k$ and $v \equiv c\, e'_1 \ldots e'_k$ is a value, then

$$\begin{array}{lcl} [\![e]\!]^\sharp & \supseteq & (v \in [\![e_0]\!]^\sharp)\ ?\ [\![e_i]\!]^\sharp \ : \ \emptyset \\ [\![y_j]\!]^\sharp & \supseteq & (v \in [\![e_0]\!]^\sharp)\ ?\ [\![e'_j]\!]^\sharp \ : \ \emptyset \end{array}$$

  If $p_i \equiv (y_1, \ldots, y_k)$ and $v \equiv (e'_1, \ldots, e'_k)$ is a value, then

$$\begin{array}{lcl} [\![e]\!]^\sharp & \supseteq & (v \in [\![e_0]\!]^\sharp)\ ?\ [\![e_i]\!]^\sharp \ : \ \emptyset \\ [\![y_j]\!]^\sharp & \supseteq & (v \in [\![e_0]\!]^\sharp)\ ?\ [\![e'_j]\!]^\sharp \ : \ \emptyset \end{array}$$

  If $p_i \equiv y$, then

$$\begin{array}{lcl} [\![e]\!]^\sharp & \supseteq & [\![e_i]\!]^\sharp \\ [\![y]\!]^\sharp & \supseteq & [\![e_0]\!]^\sharp \end{array}$$

Example The append-Function

Consider the concatenation of two lists. In Ocaml, we would write:

$$\begin{array}{llllll} \textbf{let rec}\ \mathsf{app} & = & \textbf{fun}\ x & \to & \textbf{match}\ x\ \textbf{with} & \\ & & & & [\,] \quad \to \ \textbf{fun}\ y \ \to \ y & \\ & & & & \mid h :: t \ \to \ \textbf{fun}\ y \ \to \ h :: \mathsf{app}\ t\ y & \\ \textbf{in}\ \mathsf{app}\ [1;2]\ [3] & & & & & \end{array}$$

The analysis then results in:

$$\begin{array}{lll} [\![\mathsf{app}]\!]^\sharp & = & \{\textbf{fun}\ x \ \to \ \textbf{match}\ldots\} \\ [\![x]\!]^\sharp & = & \{[1;2], [2], [\,]\} \\ [\![\textbf{match}\ldots]\!]^\sharp & = & \{\textbf{fun}\ y \to y, \textbf{fun}\ y \to h :: \mathsf{app}\ldots\} \\ [\![y]\!]^\sharp & = & \{[3]\} \\ \ldots & & \end{array}$$

$$
\begin{array}{lcl}
\dots & & \\
[\![h]\!]^\sharp & = & \{1,2\} \\
[\![t]\!]^\sharp & = & \{[2],[\,]\} \\
[\![\mathsf{app}\ t]\!]^\sharp & = & \\
[\![\mathsf{app}\ [1;2]]\!]^\sharp & = & \{\mathbf{fun}\ y \to y, \mathbf{fun}\ y \to h :: \mathsf{app} \dots\} \\
[\![\mathsf{app}\ t\ y]\!]^\sharp & = & \\
[\![\mathsf{app}\ [1;2]\ [3]]\!]^\sharp & = & \{[3], h :: \mathsf{app} \dots\}
\end{array}
$$

Values $c\, e_1 \dots e_k$, $(e_1, \dots, e_k)$ or operator applications $e_1 \Box e_2$ now are interpreted as recursive calls $c\, [\![e_1]\!]^\sharp \dots [\![e_k]\!]^\sharp$, $([\![e_1]\!]^\sharp, \dots, [\![e_k]\!]^\sharp)$ or $[\![e_1]\!]^\sharp \Box [\![e_2]\!]^\sharp$, respectively.

$\Longrightarrow$ regular tree grammar

... in the Example:

We obtain for $\quad A = [\![\mathsf{app}\ t\ y]\!]^\sharp$:

$$\begin{array}{lcl} A & \to & [3] \quad | \quad [\![h]\!]^\sharp :: A \\ [\![h]\!]^\sharp & \to & 1 \quad | \quad 2 \end{array}$$

Let $\mathcal{L}(e)$ denote the set of terms derivable from $[\![e]\!]^\sharp$ w.r.t. the regular tree grammar. Thus, e.g.,

$$\begin{array}{lcl} \mathcal{L}(h) & = & \{1, 2\} \\ \mathcal{L}(\mathsf{app}\ t\ y) & = & \{[a_1; \ldots, a_r; 3] \mid r \geq 0, a_i \in \{1, 2\}\} \end{array}$$

4.3 An Operational Semantics

Idea:

We construct a Big-Step operational semantics which evaluates expressions w.r.t. an environment :-)

Values are of the form:

$$v ::= b \mid c\, v_1 \ldots c_k \mid (v_1, \ldots, v_k) \mid (\textbf{fun}\, x \rightarrow e, \eta)$$

Examples for Values:

$$c\, 1$$
$$[1;2] = ::\, 1\, (::\, 2\, [\,])$$
$$(\textbf{fun}\, x \rightarrow x{::}y, \{y \mapsto [5]\})$$

Expressions are evaluated w.r.t. an environment $\eta : \mathit{Vars} \to \mathit{Values}$.

The Big-Step operational semantics provides rules to infer the value to which an expression is evaluated w.r.t. a given environment, i.e., deals with statements of the form:

$$(e, \eta) \Longrightarrow v$$

Values:

$$(b, \eta) \Longrightarrow b$$

$$(\mathbf{fun}\, x \to e, \eta) \Longrightarrow (\mathbf{fun}\, x \to e, \eta)$$

$$\frac{(e_1, \eta) \Longrightarrow v_1 \quad \ldots \quad (e_k, \eta) \Longrightarrow v_k}{(c\, e_1 \ldots e_k, \eta) \Longrightarrow c\, v_1 \ldots v_k}$$

$$\frac{(e_1, \eta) \Longrightarrow v_1 \quad \ldots \quad (e_k, \eta) \Longrightarrow v_k}{((e_1, \ldots, e_k), \eta) \Longrightarrow (v_1, \ldots, v_k)}$$

Global Definition:

$$\frac{\begin{array}{l} \textbf{let rec} \ \ldots \ x = e \ \ldots \ \textbf{in} \ \ldots \\ (e, \emptyset) \Longrightarrow v \end{array}}{(x, \eta) \Longrightarrow v}$$

Function Application:

$$\frac{(e_1, \eta) \Longrightarrow (\textbf{fun}\ x\ \rightarrow\ e, \eta_1) \quad (e_2, \eta) \Longrightarrow v_2 \quad (e, \eta_1 \oplus \{x \mapsto v_2\}) \Longrightarrow v_3}{(e_1\ e_2, \eta) \Longrightarrow v_3}$$