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- $\mathsf{dec}\,x = x - 1$ is monotonic.
- $\mathsf{inv}\,x = -x$ is not monotonic :-)

Theorem:

If $f_1 : \mathbb{D}_1 \to \mathbb{D}_2$ and $f_2 : \mathbb{D}_2 \to \mathbb{D}_3$ are monotonic, then also $f_2 \circ f_1 : \mathbb{D}_1 \to \mathbb{D}_3$:-)

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If $\mathbb{D}_2$ is a complete lattice, then the set $[\mathbb{D}_1 \to \mathbb{D}_2]$ of monotonic functions $f : \mathbb{D}_1 \to \mathbb{D}_2$ is also a complete lattice where

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In particular for $F \subseteq [\mathbb{D}_1 \to \mathbb{D}_2]$,

$$\bigsqcup F = f \quad \text{mit} \quad f\,x = \bigsqcup \{g\,x \mid g \in F\}$$

For functions $f_i\,x = a_i \cap x \cup b_i$, the operations "$\circ$", "$\sqcup$" and "$\sqcap$" can be explicitly defined by:

$$
\begin{aligned}
(f_2 \circ f_1)\,x &= \boxed{a_1 \cap a_2} \cap x \cup \boxed{a_2 \cap b_1 \cup b_2} \\
(f_1 \sqcup f_2)\,x &= \boxed{(a_1 \cup a_2)} \cap x \cup \boxed{b_1 \cup b_2} \\
(f_1 \sqcap f_2)\,x &= \boxed{(a_1 \cup b_1) \cap (a_2 \cup b_2)} \cap x \cup \boxed{b_1 \cap b_2}
\end{aligned}
$$

Wanted: minimally small solution for:

$$x_i \sqsupseteq f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n \qquad (*)$$

where all $f_i : \mathbb{D}^n \to \mathbb{D}$ are monotonic.

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- If all f_i are monotonic, then also F :-)
- We successively approximate a solution. We construct:

  $$\underline{\perp}, \quad F\,\underline{\perp}, \quad F^2\,\underline{\perp}, \quad F^3\,\underline{\perp}, \quad \ldots$$

Hope: We eventually reach a solution ... ???

Example: $\mathbb{D} = 2^{\{a,b,c\}}, \quad \sqsubseteq \, = \, \subseteq$

$$
\begin{array}{lll}
x_1 & \supseteq & \{a\} \cup x_3 \\
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The Iteration:

	0	1	2	3	4
x_1	$\emptyset$				
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Theorem

- $\underline{\bot}, F\,\underline{\bot}, F^2\,\underline{\bot}, \ldots$ form an ascending chain :

$$\underline{\bot} \quad \sqsubseteq \quad F\,\underline{\bot} \quad \sqsubseteq \quad F^2\,\underline{\bot} \quad \sqsubseteq \quad \ldots$$

- If $F^k\,\underline{\bot} = F^{k+1}\,\underline{\bot}$, a solution is obtained which is the least one :-)
- If all ascending chains are finite, such a k always exists.

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Proof

The first claim follows by complete induction:

Foundation: $F^0\,\underline{\perp} = \underline{\perp} \sqsubseteq F^1\,\underline{\perp}$:-)

Step: Assume $F^{i-1}\,\underline{\bot} \;\sqsubseteq\; F^{i}\,\underline{\bot}$. Then

$$F^{i}\,\underline{\bot} = F\,(F^{i-1}\,\underline{\bot}) \sqsubseteq F\,(F^{i}\,\underline{\bot}) = F^{i+1}\,\underline{\bot}$$

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Conclusion:

If $\mathbb{D}$ is finite, a solution can be found which is definitely the least :-)

Question:

What, if $\mathbb{D}$ is not finite ???

Theorem — Knaster – Tarski

Assume $\mathbb{D}$ is a complete lattice. Then every monotonic function $f : \mathbb{D} \to \mathbb{D}$ has a least fixpoint $d_0 \in \mathbb{D}$.

Let $P = \{d \in \mathbb{D} \mid f\, d \sqsubseteq d\}$.

Then $d_0 = \bigsqcap P$.

Bronisław Knaster (1893-1980), topology

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Then $d_0 = \bigsqcap P$.

Proof:

(1) $d_0 \in P$:

$$f\,d_0 \sqsubseteq f\,d \sqsubseteq d \quad \text{for all } d \in P$$

$$\Longrightarrow \quad f\,d_0 \quad \text{is a lower bound of } P$$

$$\Longrightarrow \quad f\,d_0 \sqsubseteq d_0 \quad \text{since } d_0 = \bigsqcap P$$

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(3) d_0 is least fixpoint:

$$f\, d_1 = d_1 \sqsubseteq d_1 \quad \text{an other fixpoint}$$

$$\Longrightarrow \quad d_1 \in P$$

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The least fixpoint d_0 is in P and a lower bound :-)

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Assume $x_i \sqsupseteq f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n$ $(*)$

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$\Longrightarrow$ least solution of $(*)$ $==$ least fixpoint of F :-)

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Assume $f\,x = x + 1$. Then

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$\Longrightarrow$ Ordinary iteration will never reach a fixpoint :-(

$\Longrightarrow$ Sometimes, transfinite iteration is needed :-)

Conclusion:

Systems of inequations can be solved through fixpoint iteration, i.e., by repeated evaluation of right-hand sides   :-)

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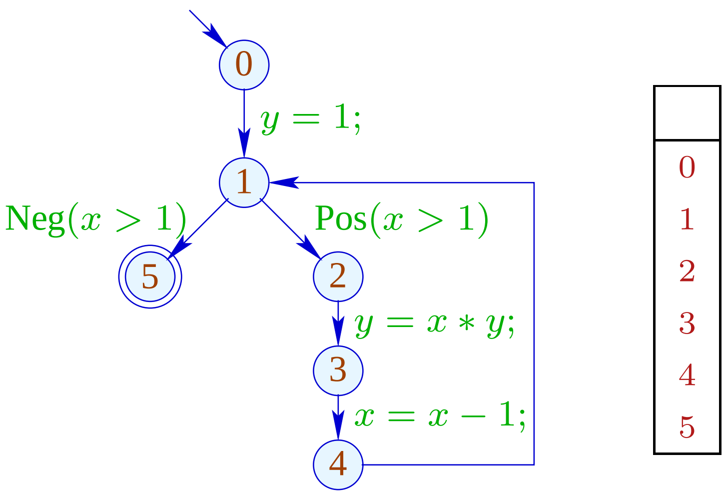

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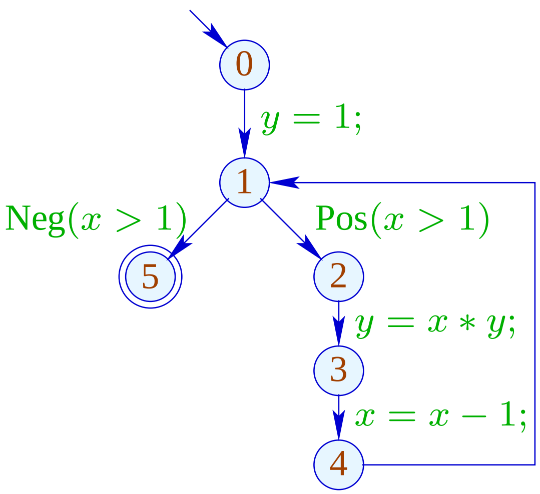

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0	$\emptyset$
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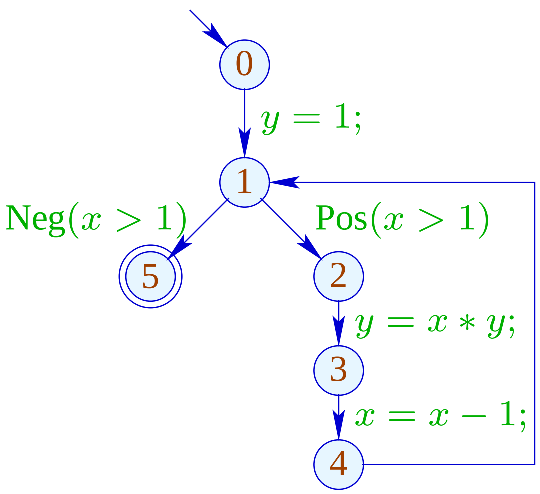

	1	2
0	$\emptyset$	$\emptyset$
1	$\{1, x > 1, x - 1\}$	$\{1\}$
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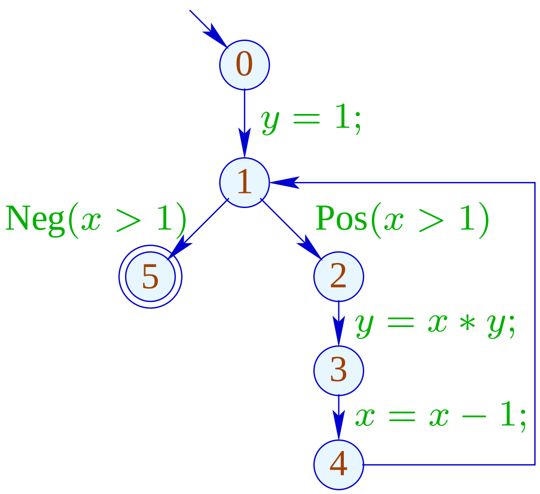

	1	2	3
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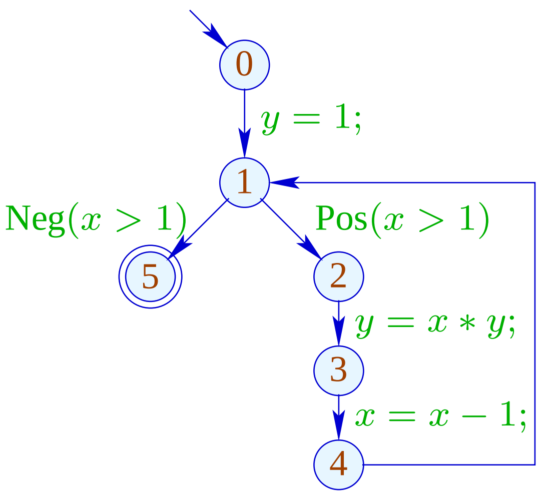

	1	2	3	4
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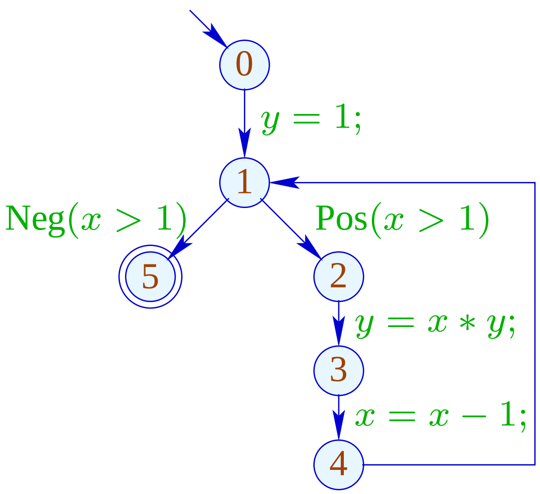

	1	2	3	4	5
0	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	
1	$\{1, x > 1, x - 1\}$	$\{1\}$	$\{1\}$	$\{1\}$	
2	$Expr$	$\{1, x > 1, x - 1\}$	$\{1, x > 1\}$	$\{1, x > 1\}$	
3	$\{1, x > 1, x - 1\}$	$\{1, x > 1, x - 1\}$	$\{1, x > 1, x - 1\}$	$\{1, x > 1\}$	dito
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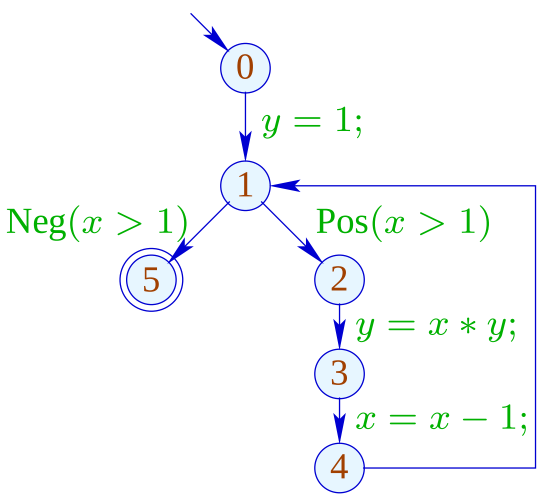

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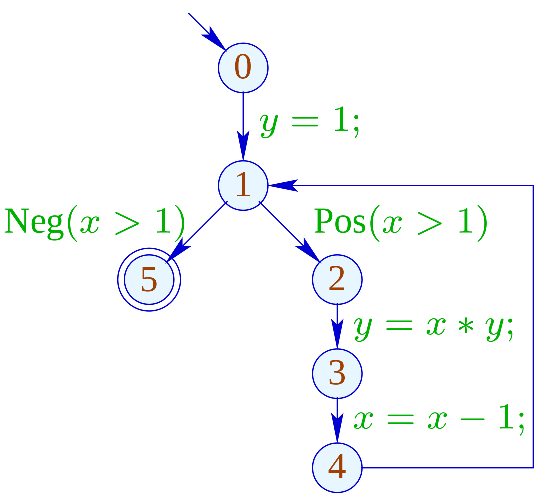

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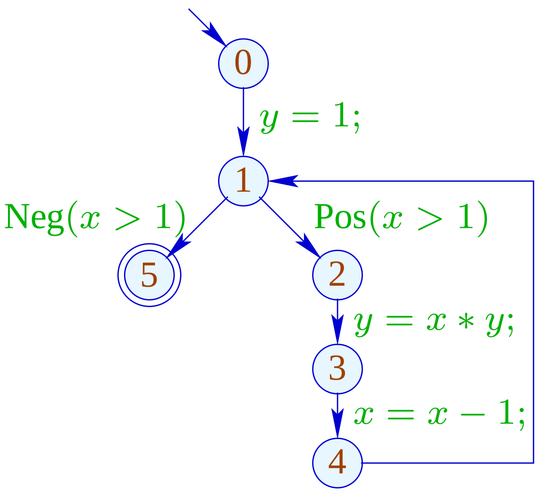

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The code for Round Robin Iteration in Java looks as follows:

```
for (i = 1; i ≤ n; i++) x_i = ⊥;
do {
        finished = true;
        for (i = 1; i ≤ n; i++) {
                new = f_i(x_1, ..., x_n);
                if (!(x_i ⊒ new)) {
                        finished = false;
                        x_i = x_i ⊔ new;
                }
        }
} while (!finished);
```

Correctness:

Assume $y_i^{(d)}$ is the i-th component of $F^d \underline{\perp}$.

Assume $x_i^{(d)}$ is the value of x_i after the d-th RR-iteration.

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(2) $x_i^{(d)} \sqsubseteq z_i$ for every solution $(z_1, \ldots, z_n)$:-)

(3) If RR-iteration terminates after d rounds, then
$(x_1^{(d)}, \ldots, x_n^{(d)})$ is a solution :-))

Caveat:

The efficiency of RR-iteration depends on the ordering of the unknowns !!!

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Good:

$\rightarrow$ u before v, if $u \rightarrow^* v$;

$\rightarrow$ entry condition before loop body :-)

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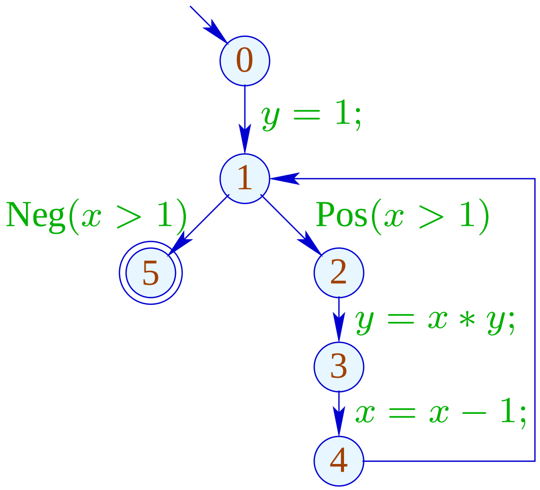
$\rightarrow$ u before v, if $u \rightarrow^* v$;

$\rightarrow$ entry condition before loop body :-)

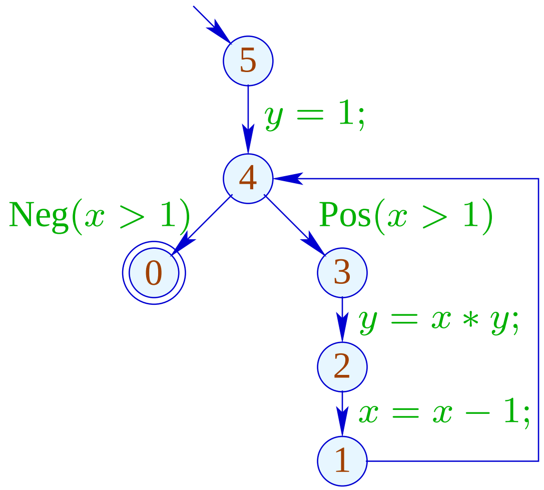
Bad:

e.g., post-order DFS of the CFG, starting at start :-)

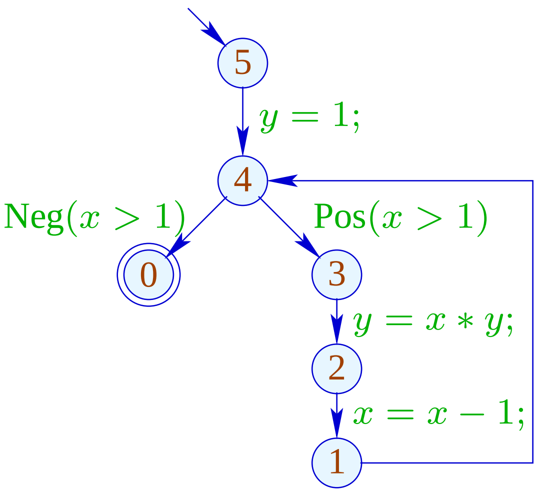
Good:



Bad:

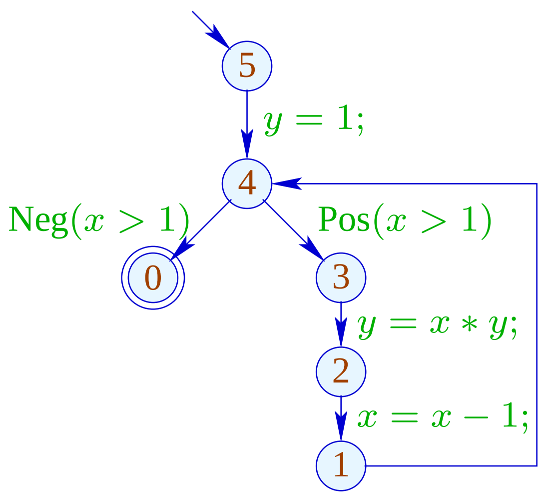


Inefficient Round Robin Iteration:



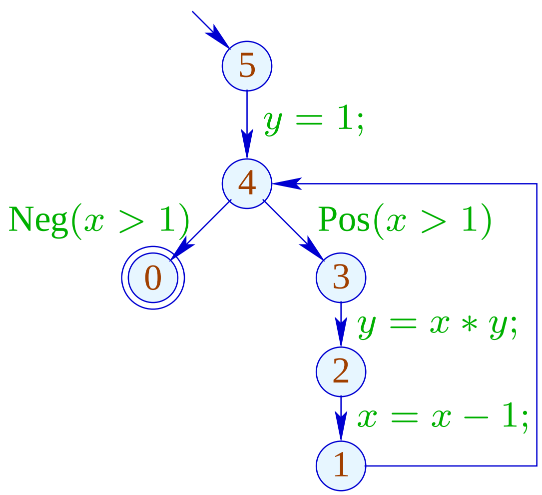
0
1
2
3
4
5

Inefficient Round Robin Iteration:



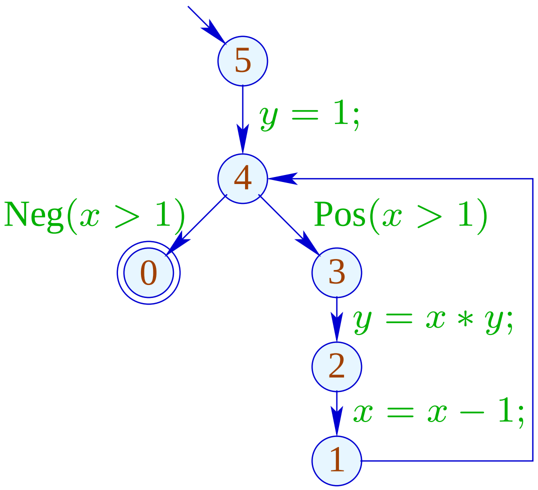
	1
0	$Expr$
1	$\{1\}$
2	$\{1, x - 1, x > 1\}$
3	$Expr$
4	$\{1\}$
5	$\emptyset$

Inefficient Round Robin Iteration:



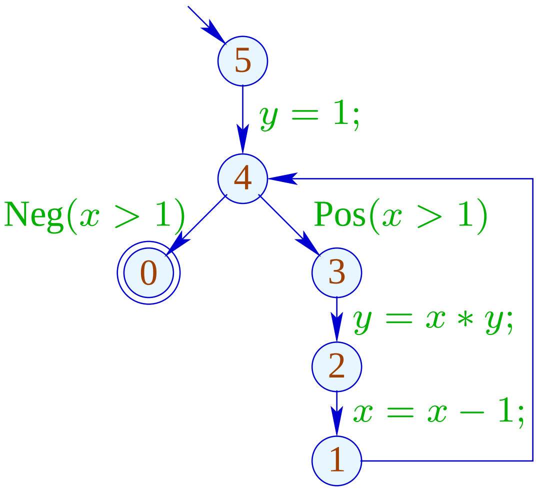
	1	2
0	$Expr$	$\{1, x > 1\}$
1	$\{1\}$	$\{1\}$
2	$\{1, x - 1, x > 1\}$	$\{1, x - 1, x > 1\}$
3	$Expr$	$\{1, x > 1\}$
4	$\{1\}$	$\{1\}$
5	$\emptyset$	$\emptyset$

Inefficient Round Robin Iteration:



	1	2	3
0	$Expr$	$\{1, x > 1\}$	$\{1, x > 1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{1, x - 1, x > 1\}$	$\{1, x - 1, x > 1\}$	$\{1, x > 1\}$
3	$Expr$	$\{1, x > 1\}$	$\{1, x > 1\}$
4	$\{1\}$	$\{1\}$	$\{1\}$
5	$\emptyset$	$\emptyset$	$\emptyset$

Inefficient Round Robin Iteration:



	1	2	3	4
0	$Expr$	$\{1, x > 1\}$	$\{1, x > 1\}$	
1	$\{1\}$	$\{1\}$	$\{1\}$	
2	$\{1, x - 1, x > 1\}$	$\{1, x - 1, x > 1\}$	$\{1, x > 1\}$	dito
3	$Expr$	$\{1, x > 1\}$	$\{1, x > 1\}$	
4	$\{1\}$	$\{1\}$	$\{1\}$	
5	$\emptyset$	$\emptyset$	$\emptyset$	

$\Longrightarrow$ significantly less efficient :-)

... end of background on: Complete Lattices