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For a complete lattice $\mathbb{D}$, consider systems:

$$\begin{array}{lll} \mathcal{I}[start] & \sqsupseteq & d_0 \\ \mathcal{I}[v] & \sqsupseteq & [\![k]\!]^\sharp\,(\mathcal{I}[u]) \qquad k = (u, \_, v) \quad \text{edge} \end{array}$$

where $d_0 \in \mathbb{D}$ and all $[\![k]\!]^\sharp : \mathbb{D} \to \mathbb{D}$ are monotonic ...

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$\Longrightarrow$ Monotonic Analysis Framework

Wanted: MOP (Merge Over all Paths)

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Jeffrey D. Ullman, Stanford

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$$\mathcal{I}[v] \sqsupseteq \mathcal{I}^*[v] \qquad \text{for every} \quad v$$

In particular: $\mathcal{I}[v] \sqsupseteq [\![\pi]\!]^\sharp \, d_0$ for every $\pi : \mathit{start} \to^* v$

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Then:

$$\begin{array}{llll}
 & [\![\pi']\!]^\sharp\, d_0 & \sqsubseteq & \mathcal{I}[u] & \text{by I.H. for } \pi \\
\Longrightarrow & [\![\pi]\!]^\sharp\, d_0 & = & [\![k]\!]^\sharp\, ([\![\pi']\!]^\sharp\, d_0) & \\
 & & \sqsubseteq & [\![k]\!]^\sharp\, (\mathcal{I}[u]) & \text{since } [\![k]\!]^\sharp \text{ monotonic} \\
 & & \sqsubseteq & \mathcal{I}[v] & \text{since } \mathcal{I} \text{ solution} \quad \text{:-))}
\end{array}$$

Disappointment:

Are solutions of the constraint system just upper bounds ???

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In general: yes :-(

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With the notable exception when all functions $[\![k]\!]^\sharp$ are distributive ...
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The function $f : \mathbb{D}_1 \to \mathbb{D}_2$ is called

- distributive, if $f\,(\bigsqcup X) = \bigsqcup\{f\,x \mid x \in X\}$ for all $\emptyset \neq X \subseteq \mathbb{D}$;
- strict, if $f\,\bot = \bot$.
- totally distributive, if f is distributive and strict.

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  Distributivity:

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\begin{aligned}
f\,(x_1 \cup x_2) &= a \cap (x_1 \cup x_2) \cup b \\
&= a \cap x_1 \cup a \cap x_2 \cup b \\
&= f\,x_1 \cup f\,x_2 \qquad \text{:-)}
\end{aligned}
$$

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  Distributivity:

$$\begin{array}{lllll} f\,((1,4) \sqcup (4,1)) & = & f\,(4,4) & = & 8 \\ & \neq & 5 & = & f\,(1,4) \sqcup f\,(4,1) \end{array}$$

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Remark:

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From that follows:

$$\begin{array}{lrcl} & f\,b & = & f\,(a \sqcup b) \\ & & = & f\,a \sqcup f\,b \\ \Longrightarrow & f\,a & \sqsubseteq & f\,b \end{array}$$

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Then:

Theorem

Kildall 1972

If all effects of edges $[\![k]\!]^\sharp$ are distributive, then: $\mathcal{I}^*[v] = \mathcal{I}[v]$
for all v .

Gary A. Kildall (1942-1994).

Has developed the operating system CP/M and GUIs for PCs.

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Proof:

It suffices to prove that $\mathcal{I}^*$ is a solution :-)

For this, we show that $\mathcal{I}^*$ satisfies all constraints :-))

(1) We prove for *start* :

$$\begin{aligned} \mathcal{I}^*[start] &= \bigsqcup\{[\![\pi]\!]^\sharp\, d_0 \mid \pi : start \to^* start\} \\ &\sqsupseteq [\![\epsilon]\!]^\sharp\, d_0 \\ &\sqsupseteq d_0 \qquad \text{:-)} \end{aligned}$$

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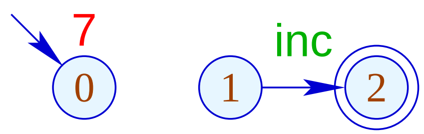
(2) For every $k = (u, \_, v)$ we prove:

$$\begin{array}{lll} \mathcal{I}^*[v] & = & \bigsqcup\{[\![\pi]\!]^\sharp\, d_0 \mid \pi : start \to^* v\} \\ & \sqsupseteq & \bigsqcup\{[\![\pi' k]\!]^\sharp\, d_0 \mid \pi' : start \to^* u\} \\ & = & \bigsqcup\{[\![k]\!]^\sharp\, ([\![\pi']\!]^\sharp\, d_0) \mid \pi' : start \to^* u\} \\ & = & [\![k]\!]^\sharp\, (\bigsqcup\{[\![\pi']\!]^\sharp\, d_0 \mid \pi' : start \to^* u\}) \\ & = & [\![k]\!]^\sharp\, (\mathcal{I}^*[u]) \end{array}$$

since $\{\pi' \mid \pi' : start \to^* u\}$ is non-empty :-)

Caveat:

- Reachability of all program points cannot be abandoned! Consider:



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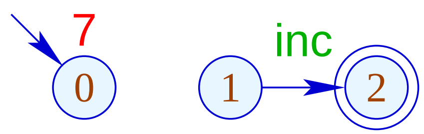

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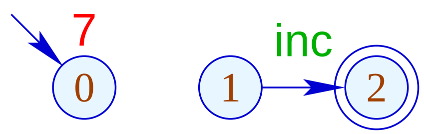

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Then:

$$\mathcal{I}[2] = \mathsf{inc}\, 0 = 1$$
$$\mathcal{I}^*[2] = \bigsqcup \emptyset = 0$$

- Unreachable program points can always be thrown away :-)

Summary and Application:

→ The effects of edges of the analysis of availability of expressions are distributive:

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\begin{aligned}
(a \cup (x_1 \cap x_2))\backslash b &= ((a \cup x_1) \cap (a \cup x_2))\backslash b \\
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→ If all effects of edges are distributive, then the MOP can be computed by means of the constraint system and RR-iteration. :-)

→ If not all effects of edges are distributive, then RR-iteration for the constraint system at least returns a safe upper bound to the MOP :-)