Division: $[l_1, u_1] \,/^\sharp\, [l_2, u_2] \;=\; [a, b]$

- If 0 is not contained in the interval of the denominator, then:

$$\begin{aligned} a &= l_1/l_2 \sqcap l_1/u_2 \sqcap u_1/l_2 \sqcap u_1/u_2 \\ b &= l_1/l_2 \sqcup l_1/u_2 \sqcup u_1/l_2 \sqcup u_1/u_2 \end{aligned}$$

- If: $l_2 \leq 0 \leq u_2$, we define:

$$[a, b] \;=\; [-\infty, +\infty]$$

Equality:

$$[l_1, u_1] ==^\sharp [l_2, u_2] \;=\; \begin{cases} [1,1] & \text{if} \quad l_1 = u_1 = l_2 = u_2 \\ [0,0] & \text{if} \quad u_1 < l_2 \vee u_2 < l_1 \\ [0,1] & \text{otherwise} \end{cases}$$

Equality:

$$[l_1, u_1] ==^\sharp [l_2, u_2] \; = \; \begin{cases} [1,1] & \text{if} \quad l_1 = u_1 = l_2 = u_2 \\ [0,0] & \text{if} \quad u_1 < l_2 \vee u_2 < l_1 \\ [0,1] & \text{otherwise} \end{cases}$$

Example:

$$\begin{aligned} [42,42] ==^\sharp [42,42] &= [1,1] \\ [0,7] ==^\sharp [0,7] &= [0,1] \\ [1,2] ==^\sharp [3,4] &= [0,0] \end{aligned}$$

Less:

$$[l_1, u_1] \;<^\sharp\; [l_2, u_2] \;=\; \begin{cases} [1,1] & \text{if} \quad u_1 < l_2 \\ [0,0] & \text{if} \quad u_2 \leq l_1 \\ [0,1] & \text{otherwise} \end{cases}$$

Less:

$$[l_1, u_1] <^\sharp [l_2, u_2] = \begin{cases} [1,1] & \text{if} \quad u_1 < l_2 \\ [0,0] & \text{if} \quad u_2 \leq l_1 \\ [0,1] & \text{otherwise} \end{cases}$$

Example:

$$\begin{aligned} [1,2] <^\sharp [9,42] &= [1,1] \\ [0,7] <^\sharp [0,7] &= [0,1] \\ [3,4] <^\sharp [1,2] &= [0,0] \end{aligned}$$

By means of $\mathbb{I}$ we construct the complete lattice:

$$\mathbb{D}_{\mathbb{I}} = (\,Vars \to \mathbb{I})_{\perp}$$

Description Relation:

$$\rho \;\Delta\; D \qquad \text{iff} \qquad D \neq \perp \quad \wedge \quad \forall\, x \in Vars : \; (\rho\, x) \;\Delta\; (D\; x)$$

The abstract evaluation of expressions is defined analogously to constant propagation. We have:

$$([\![e]\!]\; \rho) \;\Delta\; ([\![e]\!]^{\sharp}\; D) \qquad \text{whenever} \qquad \rho \;\Delta\; D$$

The Effects of Edges:

$$
\begin{array}{lcl}
[\![;]\!]^\sharp\, D & = & D \\
[\![x = e;]\!]^\sharp\, D & = & D \oplus \{x \mapsto [\![e]\!]^\sharp\, D\} \\
[\![x = M[e];]\!]^\sharp\, D & = & D \oplus \{x \mapsto \top\} \\
[\![M[e_1] = e_2;]\!]^\sharp\, D & = & D \\
[\![\mathrm{Pos}\,(e)]\!]^\sharp\, D & = & \begin{cases} \bot & \text{if} \quad [0,0] = [\![e]\!]^\sharp\, D \\ D & \text{otherwise} \end{cases} \\
[\![\mathrm{Neg}\,(e)]\!]^\sharp\, D & = & \begin{cases} D & \text{if} \quad [0,0] \sqsubseteq [\![e]\!]^\sharp\, D \\ \bot & \text{otherwise} \end{cases}
\end{array}
$$

... given that $D \neq \bot$:-)

Better Exploitation of Conditions:

$$[\![\text{Pos}\,(e)]\!]^\sharp\,D \quad = \quad \begin{cases} \bot & \text{if} \quad [0,0] = [\![e]\!]^\sharp\,D \\ D_1 & \text{otherwise} \end{cases}$$

where :

$$D_1 \quad = \quad \begin{cases} D \oplus \{x \mapsto (D\,x) \sqcap ([\![e_1]\!]^\sharp\,D)\} & \text{if } e \equiv x == e_1 \\ D \oplus \{x \mapsto (D\,x) \sqcap [-\infty, u]\} & \text{if } e \equiv x \,\leq\, e_1, [\![e_1]\!]^\sharp\,D = [\_, u] \\ D \oplus \{x \mapsto (D\,x) \sqcap [l, \infty]\} & \text{if } e \equiv x \,\geq\, e_1, [\![e_1]\!]^\sharp\,D = [l, \_] \end{cases}$$

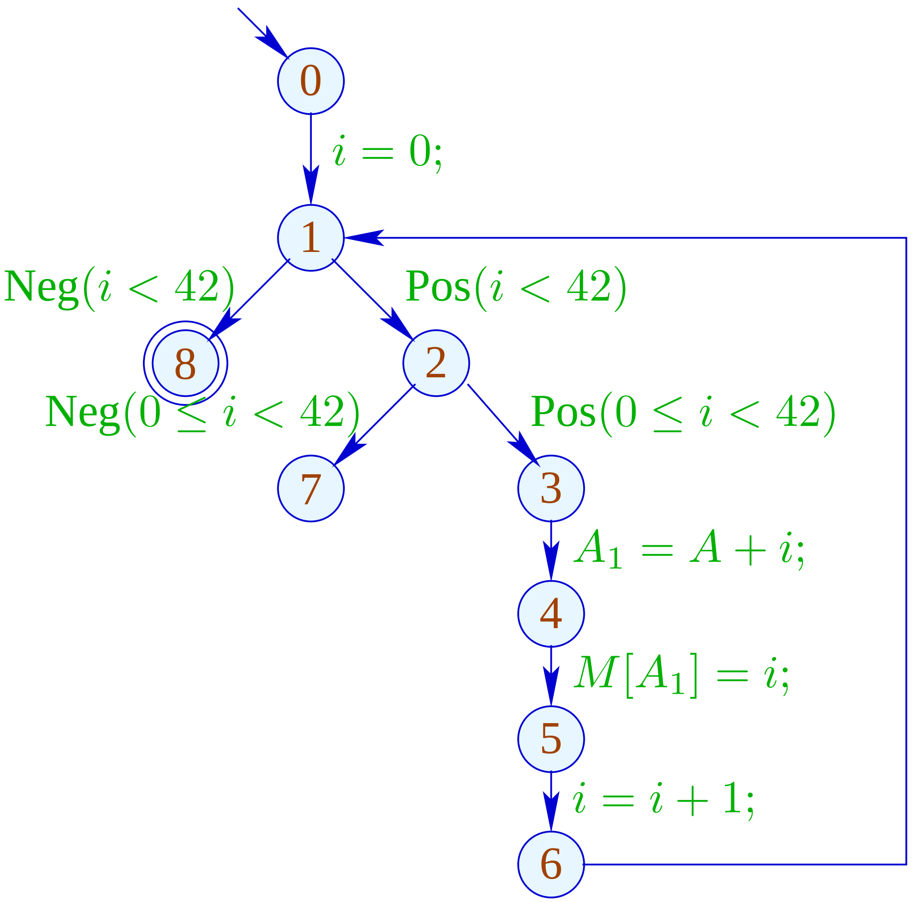
Better Exploitation of Conditions (cont.):

$$[\![\text{Neg}\,(e)]\!]^\sharp\, D \quad = \quad \begin{cases} \bot & \text{if} \quad [0,0] \not\sqsubseteq [\![e]\!]^\sharp\, D \\ D_1 & \text{otherwise} \end{cases}$$

where :

$$D_1 \quad = \quad \begin{cases} D \oplus \{x \mapsto (D\,x) \sqcap ([\![e_1]\!]^\sharp\, D)\} & \text{if } e \equiv x \neq e_1 \\ D \oplus \{x \mapsto (D\,x) \sqcap [-\infty, u]\} & \text{if } e \equiv x > e_1, [\![e_1]\!]^\sharp\, D = [\_, u] \\ D \oplus \{x \mapsto (D\,x) \sqcap [l, \infty]\} & \text{if } e \equiv x < e_1, [\![e_1]\!]^\sharp\, D = [l, \_] \end{cases}$$

Example:



	i	
	l	u
0	$-\infty$	$+\infty$
1	0	42
2	0	41
3	0	41
4	0	41
5	0	41
6	1	42
7	$\perp$	
8	42	42

Problem:

- $\rightarrow$ The solution can be computed with RR-iteration — after about 42 rounds :-(
- $\rightarrow$ On some programs, iteration may never terminate :-((

Idea 1: Widening

- Accelerate the iteration — at the prize of imprecision :-)
- Allow only a bounded number of modifications of values !!!

... in the Example:

- dis-allow updates of interval bounds in $\mathbb{Z}$ ...

$\Longrightarrow$ a maximal chain:

$$[3, 17] \sqsubset [3, +\infty] \sqsubset [-\infty, +\infty]$$

Formalization of the Approach:

Let $\quad x_i \sqsupseteq f_i(x_1, \ldots, x_n)\ , \quad i = 1, \ldots, n$ (1)

denote a system of constraints over $\mathbb{D}$ where the f_i are not necessarily monotonic.

Nonetheless, an accumulating iteration can be defined. Consider the system of equations:

$$x_i = x_i \sqcup f_i(x_1, \ldots, x_n)\ , \quad i = 1, \ldots, n \qquad (2)$$

We obviously have:

(a) $\underline{x}$ is a solution of (1) iff $\underline{x}$ is a solution of (2).

(b) The function $G : \mathbb{D}^n \to \mathbb{D}^n$ with
$G(x_1, \ldots, x_n) = (y_1, \ldots, y_n)\ , \quad y_i = x_i \sqcup f_i(x_1, \ldots, x_n)$
is increasing, i.e., $\underline{x} \sqsubseteq G\,\underline{x}$ for all $\underline{x} \in \mathbb{D}^n$.

(c) The sequence $G^k \underline{\perp}$, $k \geq 0,$ is an ascending chain:

$$\underline{\perp} \sqsubseteq G \underline{\perp} \sqsubseteq \ldots \sqsubseteq G^k \underline{\perp} \sqsubseteq \ldots$$

(d) If $G^k \underline{\perp} = G^{k+1} \underline{\perp} = \underline{y}$, then $\underline{y}$ is a solution of (1).

(e) If $\mathbb{D}$ has infinite strictly ascending chains, then (d) is not yet sufficient ...

but: we could consider the modified system of equations:

$$x_i = x_i \sqcup\!\!\!\!\sqcup\, f_i(x_1, \ldots, x_n) \ , \quad i = 1, \ldots, n \qquad (3)$$

for a binary operation widening:

$$\sqcup\!\!\!\!\sqcup\ : \mathbb{D}^2 \to \mathbb{D} \qquad \text{with} \qquad v_1 \sqcup v_2 \sqsubseteq v_1 \sqcup\!\!\!\!\sqcup\, v_2$$

(RR)-iteration for (3) still will compute a solution of (1) :-)

... for Interval Analysis:

- The complete lattice is: $\mathbb{D}_{\mathbb{I}} = (Vars \to \mathbb{I})_{\perp}$
- the widening $\sqcup$ is defined by:

$$\perp \sqcup D = D \sqcup \perp = D \qquad \text{and for} \quad D_1 \neq \perp \neq D_2:$$

$$\begin{array}{rcl} (D_1 \sqcup D_2)\, x & = & (D_1\, x) \sqcup (D_2\, x) \qquad \text{where} \\ [l_1, u_1] \sqcup [l_2, u_2] & = & [l, u] \qquad \text{with} \\ l & = & \begin{cases} l_1 & \text{if} \quad l_1 \leq l_2 \\ -\infty & \text{otherwise} \end{cases} \\ u & = & \begin{cases} u_1 & \text{if} \quad u_1 \geq u_2 \\ +\infty & \text{otherwise} \end{cases} \end{array}$$

$\Longrightarrow$ $\sqcup$ is not commutative !!!

Example:

$$
\begin{array}{rcl}
[0,2] \sqcup [1,2] & = & [0,2] \\
[1,2] \sqcup [0,2] & = & [-\infty,2] \\
[1,5] \sqcup [3,7] & = & [1,+\infty]
\end{array}
$$

→ Widening returns larger values more quickly.

→ It should be constructed in such a way that termination of iteration is guaranteed :-)

→ For interval analysis, widening bounds the number of iterations by:

$$\#points \cdot (1 + 2 \cdot \#Vars)$$

Conclusion:

- In order to determine a solution of (1) over a complete lattice with infinite ascending chains, we define a suitable widening and then solve (3) :-)

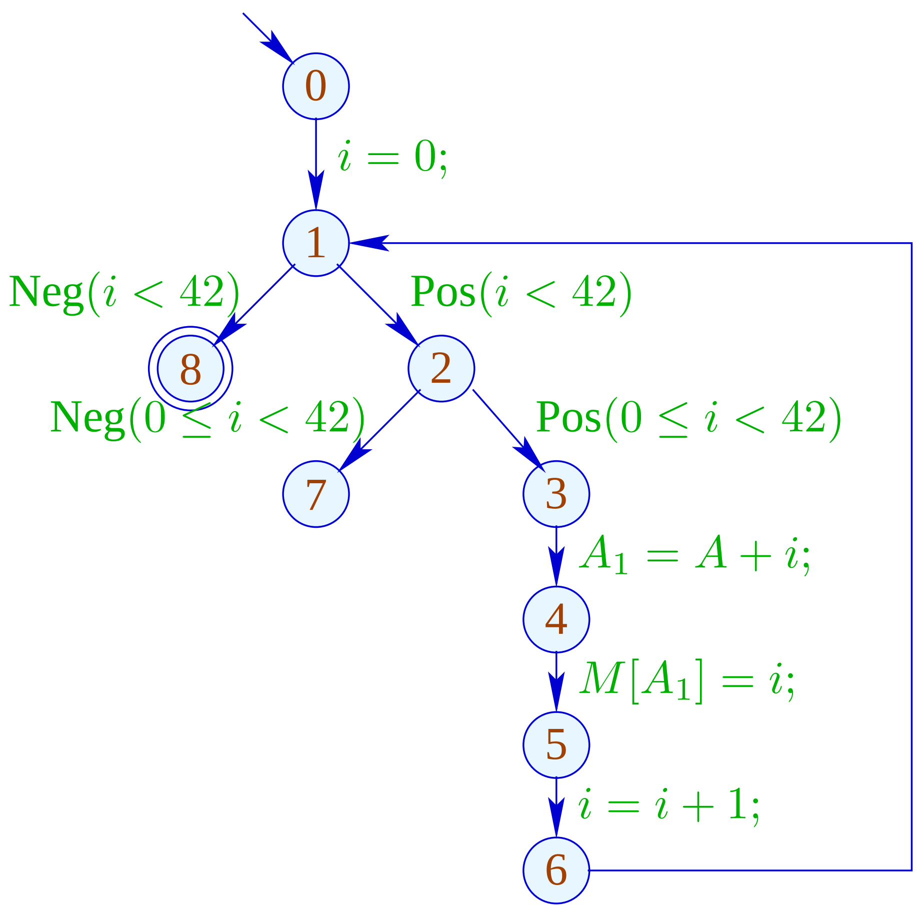
- Caveat: The construction of suitable widenings is a dark art !!!

  Often $\sqcup$ is chosen dynamically during iteration such that

  $\rightarrow$ the abstract values do not get too complicated;

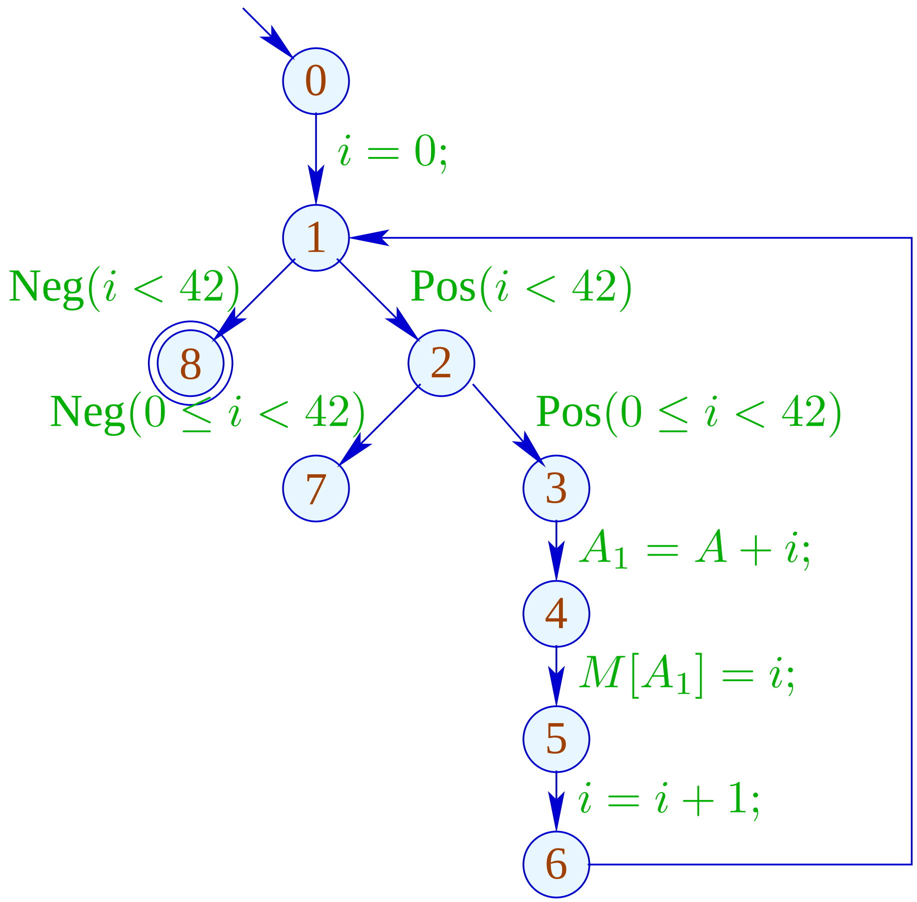
  $\rightarrow$ the number of updates remains bounded ...

Our Example:



	1	
	l	u
0	$-\infty$	$+\infty$
1	0	0
2	0	0
3	0	0
4	0	0
5	0	0
6	1	1
7	$\perp$	
8	$\perp$	

Our Example:



	1		2		3	
	l	u	l	u	l	u
0	$-\infty$	$+\infty$	$-\infty$	$+\infty$		
1	0	0	0	$+\infty$		
2	0	0	0	$+\infty$		
3	0	0	0	$+\infty$		
4	0	0	0	$+\infty$	dito	
5	0	0	0	$+\infty$		
6	1	1	1	$+\infty$		
7	$\perp$		42	$+\infty$		
8	$\perp$		42	$+\infty$		

... obviously, the result is disappointing :-(

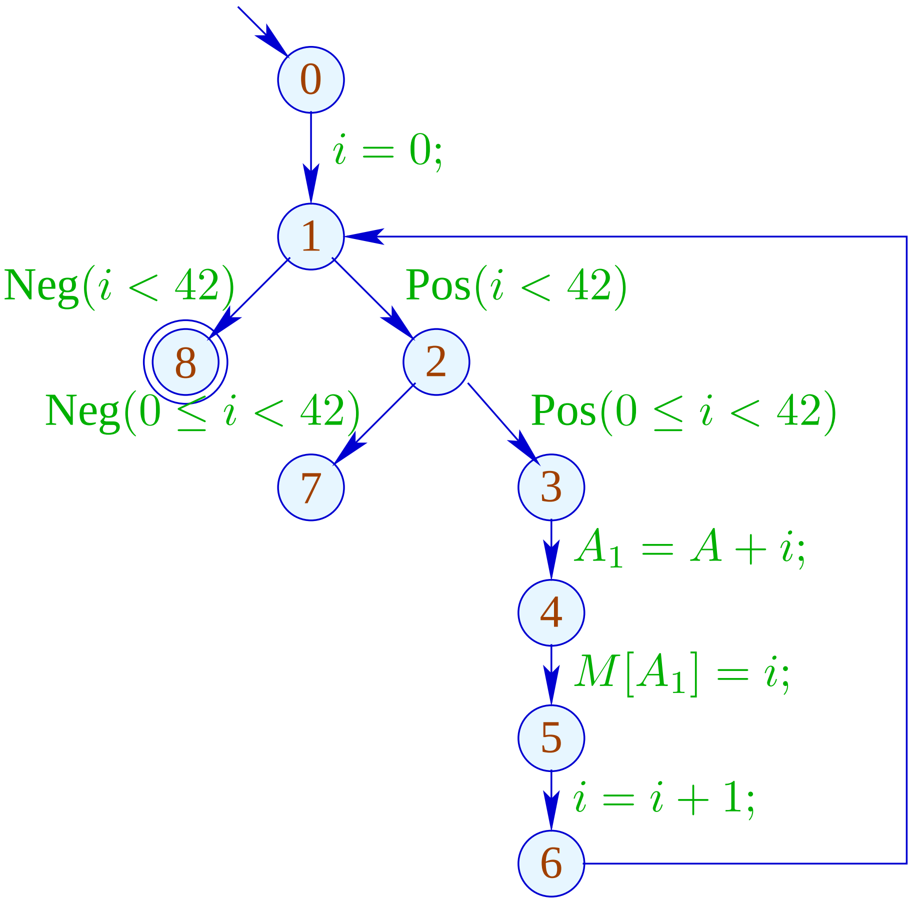
Idea 2:

In fact, acceleration with $\sqcup$ need only be applied at sufficiently many places!

A set I is a loop separator, if every loop contains at least one point from I :-)

If we apply widening only at program points from such a set I , then RR-iteration still terminates !!!

In our Example:

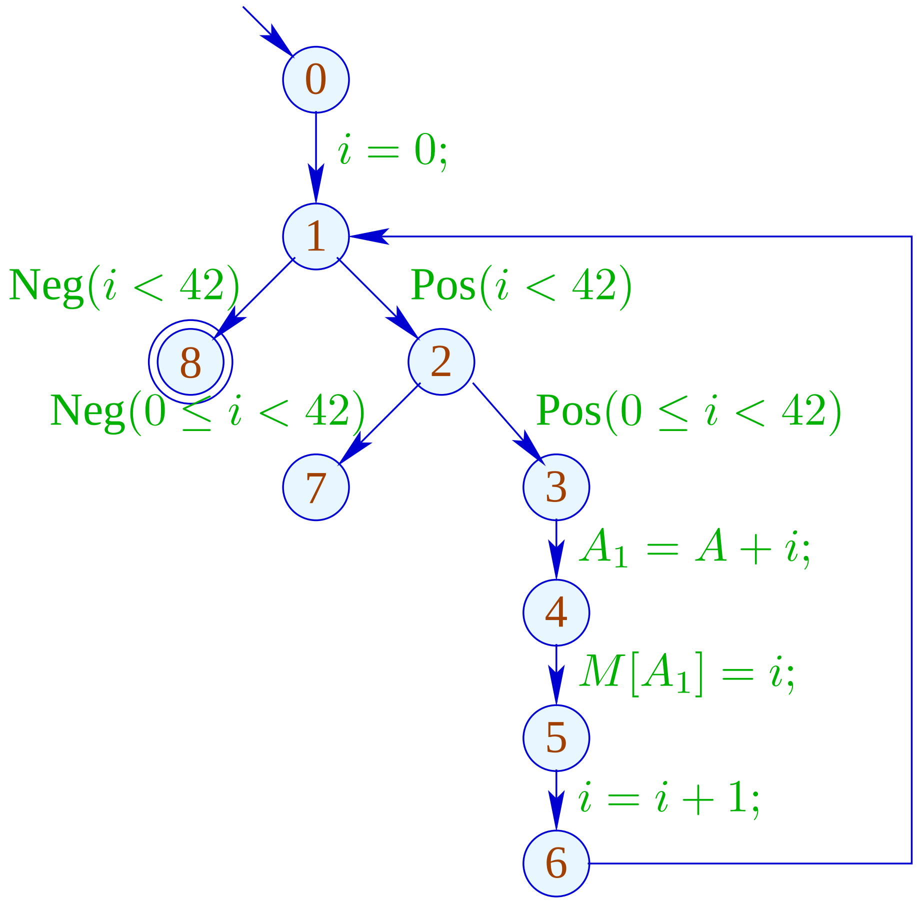


$$I_1 = \{1\}$$ or:

$$I_2 = \{2\}$$ or:

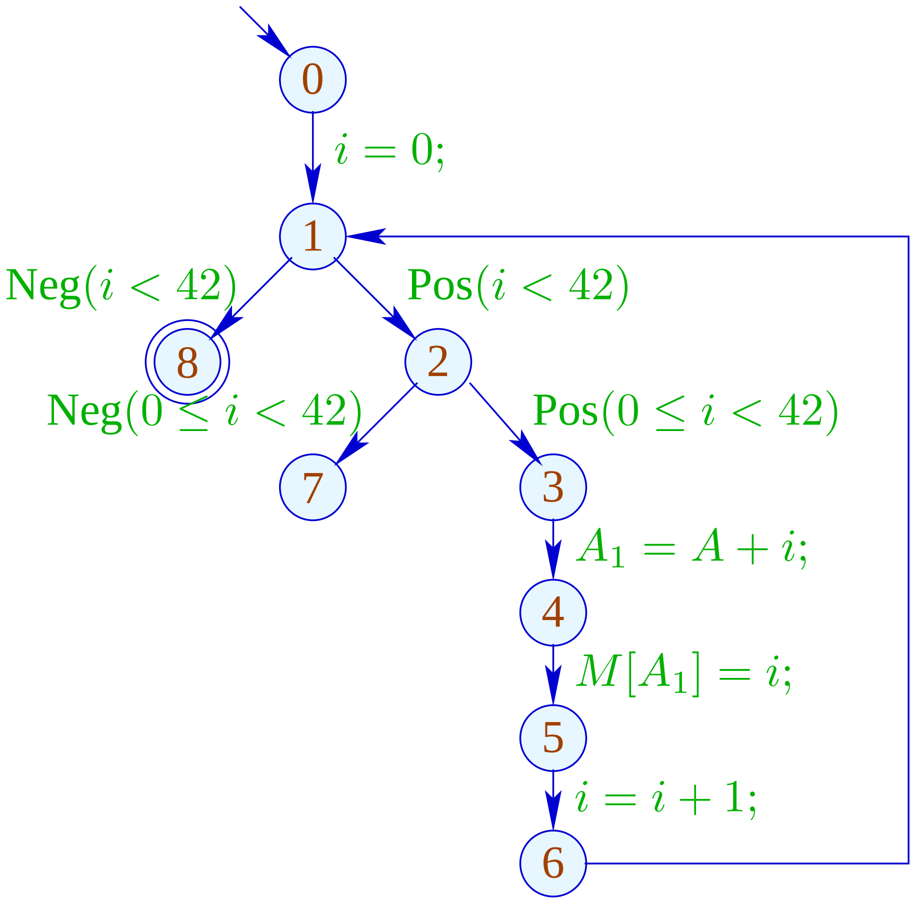
$$I_3 = \{3\}$$

The Analysis with $I = \{1\}$:



	1		2		3	
	l	u	l	u	l	u
0	$-\infty$	$+\infty$	$-\infty$	$+\infty$	dito	
1	0	0	0	$+\infty$		
2	0	0	0	41		
3	0	0	0	41		
4	0	0	0	41		
5	0	0	0	41		
6	1	1	1	42		
7	$\perp$		$\perp$			
8	$\perp$		42	$+\infty$		

The Analysis with $I = \{2\}$:



	1		2		3		4
	l	u	l	u	l	u	
0	$-\infty$	$+\infty$	$-\infty$	$+\infty$	$-\infty$	$+\infty$	dito
1	0	0	0	1	0	42	
2	0	0	0	$+\infty$	0	$+\infty$	
3	0	0	0	41	0	41	
4	0	0	0	41	0	41	
5	0	0	0	41	0	41	
6	1	1	1	42	1	42	
7	$\perp$		42	$+\infty$	42	$+\infty$	
8	$\perp$		$\perp$		42	42	

Discussion:

- Both runs of the analysis determine interesting information :-)
- The run with $I = \{2\}$ proves that always $i = 42$ after leaving the loop.
- Only the run with $I = \{1\}$ finds, however, that the outer check makes the inner check superfluous :-(

How can we find a suitable loop separator I ???

Idea 3: Narrowing

Let $\underline{x}$ denote any solution of (1), i.e.,

$$x_i \sqsupseteq f_i\,\underline{x}\;, \qquad i = 1, \ldots, n$$

Then for monotonic f_i,

$$\underline{x} \sqsupseteq F\,\underline{x} \sqsupseteq F^2\,\underline{x} \sqsupseteq \ldots \sqsupseteq F^k\,\underline{x} \sqsupseteq \ldots$$

// Narrowing Iteration

Idea 3: Narrowing

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$$x_i \sqsupseteq f_i\,\underline{x}\ , \qquad i = 1, \ldots, n$$

Then for monotonic f_i,

$$\underline{x} \sqsupseteq F\,\underline{x} \sqsupseteq F^2\,\underline{x} \sqsupseteq \ldots \sqsupseteq F^k\,\underline{x} \sqsupseteq \ldots$$

// Narrowing Iteration

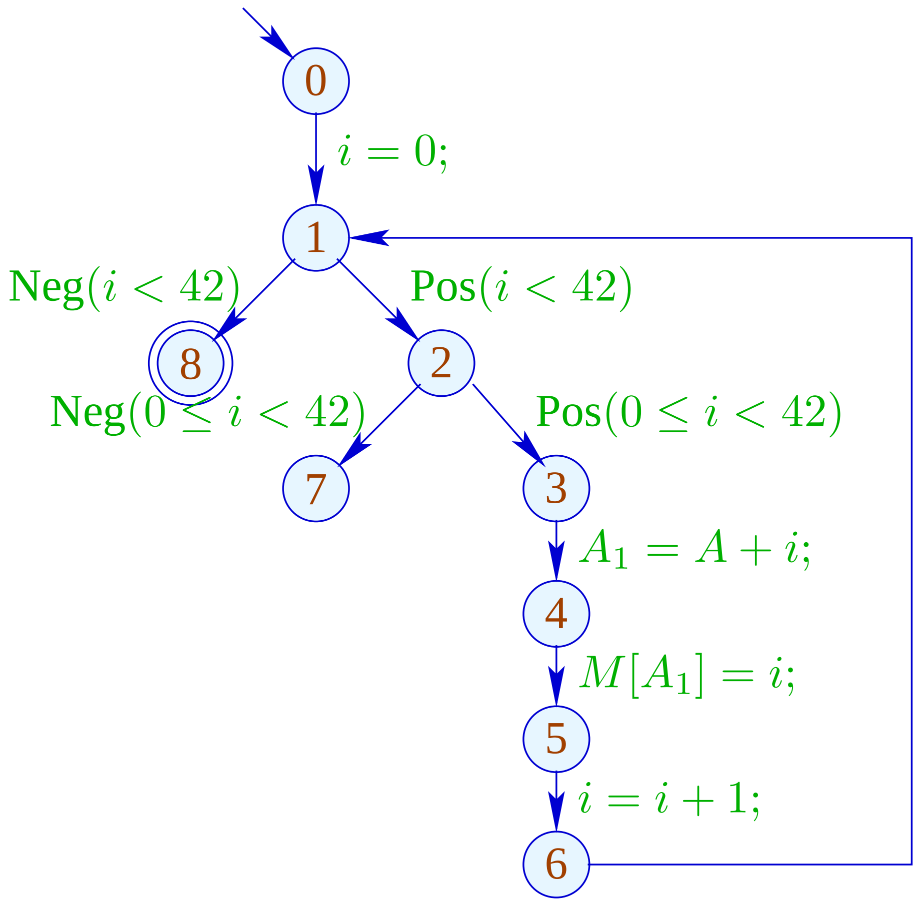
Every tuple $F^k\,\underline{x}$ is a solution of (1) :-)

$\Longrightarrow$

Termination is no problem anymore:
we stop whenever we want :-))

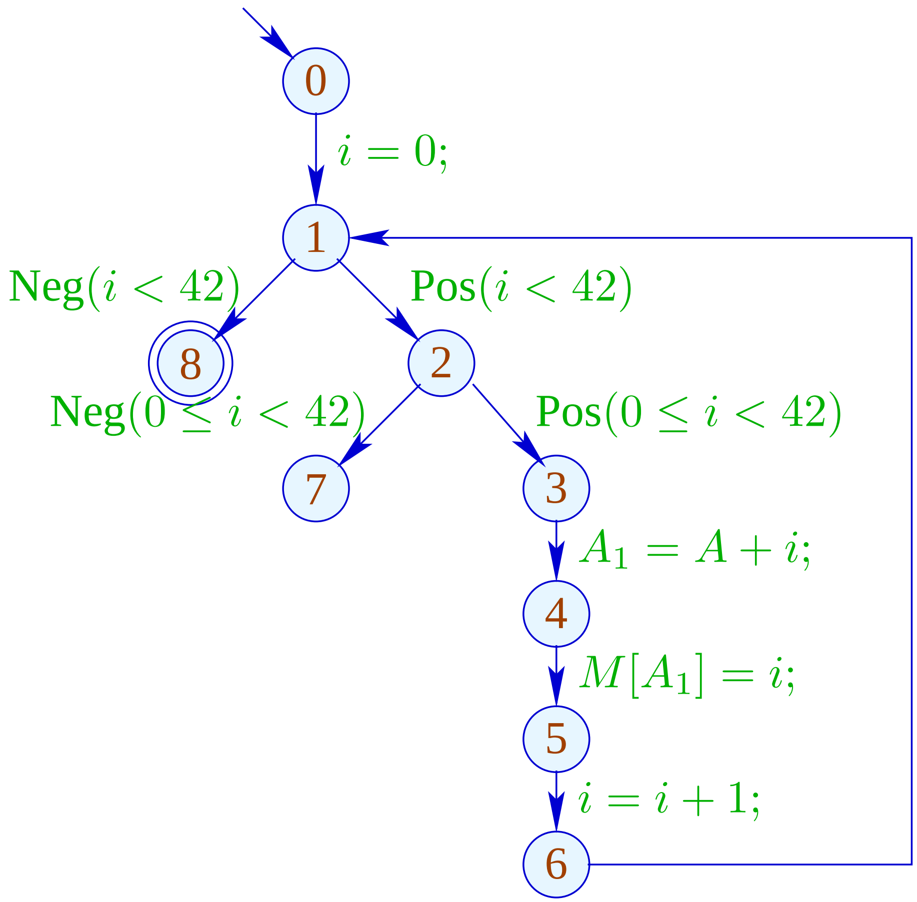
// The same also holds for RR-iteration.

Narrowing Iteration in the Example:



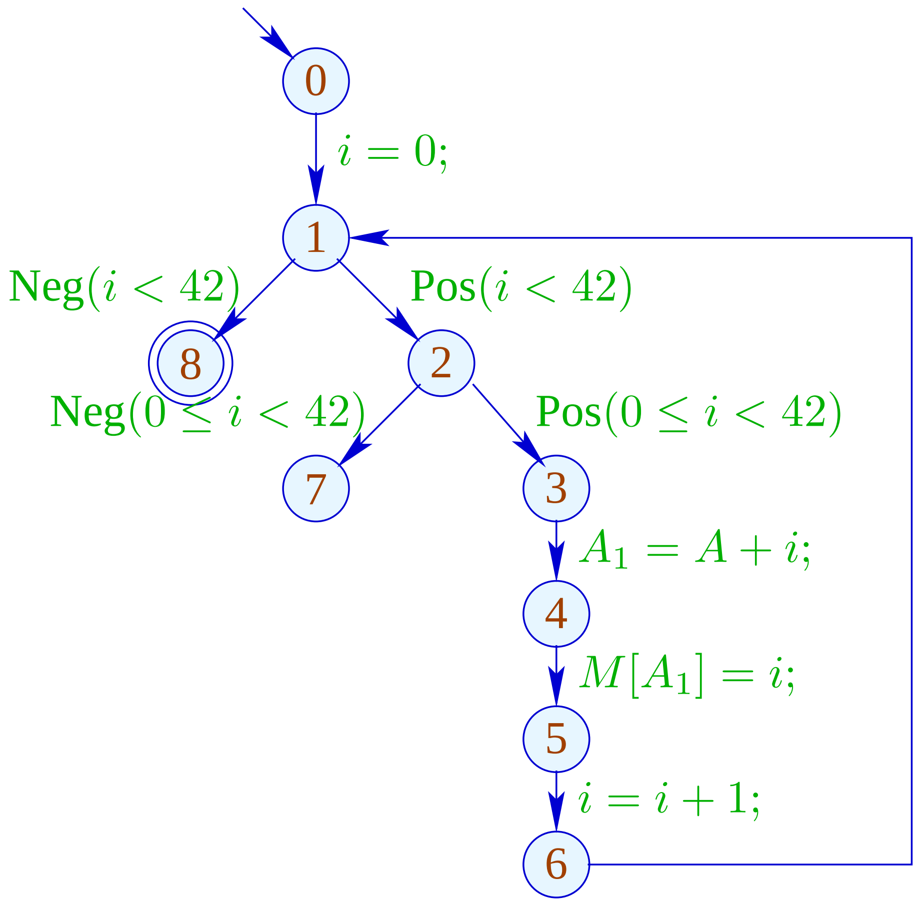
	0	
	l	u
0	$-\infty$	$+\infty$
1	0	$+\infty$
2	0	$+\infty$
3	0	$+\infty$
4	0	$+\infty$
5	0	$+\infty$
6	1	$+\infty$
7	42	$+\infty$
8	42	$+\infty$

Narrowing Iteration in the Example:



	0		1	
	l	u	l	u
0	$-\infty$	$+\infty$	$-\infty$	$+\infty$
1	0	$+\infty$	0	$+\infty$
2	0	$+\infty$	0	41
3	0	$+\infty$	0	41
4	0	$+\infty$	0	41
5	0	$+\infty$	0	41
6	1	$+\infty$	1	42
7	42	$+\infty$	$\perp$	
8	42	$+\infty$	42	$+\infty$

Narrowing Iteration in the Example:



	0		1		2	
	l	u	l	u	l	u
0	$-\infty$	$+\infty$	$-\infty$	$+\infty$	$-\infty$	$+\infty$
1	0	$+\infty$	0	$+\infty$	0	42
2	0	$+\infty$	0	41	0	41
3	0	$+\infty$	0	41	0	41
4	0	$+\infty$	0	41	0	41
5	0	$+\infty$	0	41	0	41
6	1	$+\infty$	1	42	1	42
7	42	$+\infty$	$\perp$		$\perp$	
8	42	$+\infty$	42	$+\infty$	42	42

Discussion:

- $\rightarrow$ We start with a safe approximation.
- $\rightarrow$ We find that the inner check is redundant :-)
- $\rightarrow$ We find that at exit from the loop, always $i = 42$:-))
- $\rightarrow$ It was not necessary to construct an optimal loop separator :-)))

Last Question:

Do we have to accept that narrowing may not terminate ???

4. Idea: Accelerated Narrowing

Assume that we have a solution $\underline{x} = (x_1, \ldots, x_n)$ of the system of constraints:

$$x_i \sqsupseteq f_i\,(x_1, \ldots, x_n)\;, \qquad i = 1, \ldots, n \qquad (1)$$

Then consider the system of equations:

$$x_i = x_i \sqcap f_i\,(x_1, \ldots, x_n)\;, \qquad i = 1, \ldots, n \qquad (4)$$

Obviously, we have for monotonic f_i : $H^k\,\underline{x} = F^k\,\underline{x}$:-)

where $H\,(x_1, \ldots, x_n) = (y_1, \ldots, y_n)\;, \quad y_i = x_i \sqcap f_i\,(x_1, \ldots, x_n)$.

In (4) , we replace $\sqcap$ durch by the novel operator $\overline{\sqcap}$ where:

$$a_1 \sqcap a_2 \sqsubseteq a_1 \,\overline{\sqcap}\, a_2 \sqsubseteq a_1$$

... for Interval Analysis:

We preserve finite interval bounds :-)

Therefore, $\bot \sqcap\!\!\!\!\!\sqcap D = D \sqcap\!\!\!\!\!\sqcap \bot = \bot$ and for $D_1 \neq \bot \neq D_2$:

$$
\begin{array}{rcl}
(D_1 \sqcap\!\!\!\!\!\sqcap D_2)\, x & = & (D_1\, x) \sqcap\!\!\!\!\!\sqcap (D_2\, x) \qquad \text{where} \\
{[l_1, u_1]} \sqcap\!\!\!\!\!\sqcap [l_2, u_2] & = & [l, u] \qquad \text{with} \\
l & = & \begin{cases} l_2 & \text{if} \quad l_1 = -\infty \\ l_1 & \text{otherwise} \end{cases} \\
u & = & \begin{cases} u_2 & \text{if} \quad u_1 = \infty \\ u_1 & \text{otherwise} \end{cases}
\end{array}
$$

$\Longrightarrow$ $\sqcap\!\!\!\!\!\sqcap$ is not commutative !!!