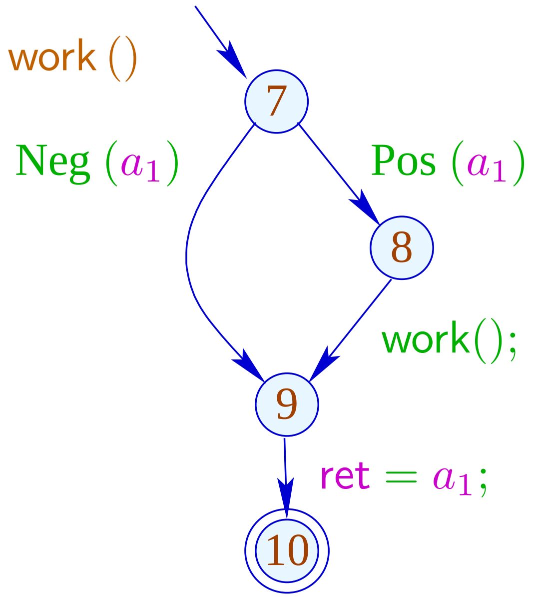
... in the Example:

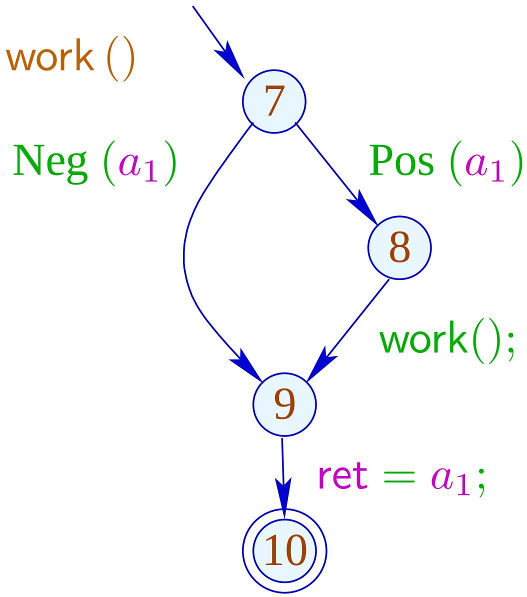
$$\begin{array}{lrcl} \text{If} & [\![\mathsf{work}]\!]^\sharp & = & \{a_1 \mapsto a_1, \mathsf{ret} \mapsto a_1, t \mapsto t\} \\ \text{then} & H\,[\![\mathsf{work}]\!]^\sharp & = & \mathsf{Id}_{\{t\}} \oplus \{a_1 \mapsto a_1, \mathsf{ret} \mapsto a_1\} \\ & & = & \{a_1 \mapsto a_1, \mathsf{ret} \mapsto a_1, t \mapsto t\} \end{array}$$

Now we can perform fixpoint iteration :-)



	1
7	$\{a_1 \mapsto a_1, \mathsf{ret} \mapsto \mathsf{ret}, t \mapsto t\}$
9	$\{a_1 \mapsto a_1, \mathsf{ret} \mapsto \mathsf{ret}, t \mapsto t\}$
10	$\{a_1 \mapsto a_1, \mathsf{ret} \mapsto a_1, t \mapsto t\}$
8	$\{a_1 \mapsto a_1, \mathsf{ret} \mapsto \mathsf{ret}, t \mapsto t\}$

$$
\begin{aligned}
[\![(8, \ldots, 9)]\!]^\sharp \circ [\![8]\!]^\sharp &= \{a_1 \mapsto a_1, \mathsf{ret} \mapsto a_1, t \mapsto t\} \circ \\
&\quad \{a_1 \mapsto a_1, \mathsf{ret} \mapsto \mathsf{ret}, t \mapsto t\} \\
&= \{a_1 \mapsto a_1, \mathsf{ret} \mapsto a_1, t \mapsto t\}
\end{aligned}
$$



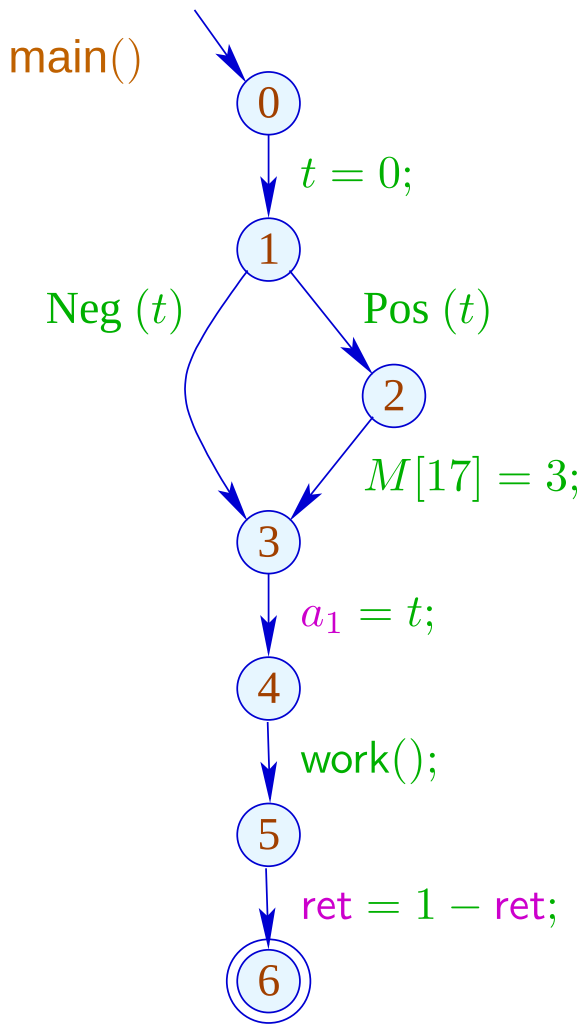
	2
7	$\{a_1 \mapsto a_1, \mathsf{ret} \mapsto \mathsf{ret}, t \mapsto t\}$
9	$\{a_1 \mapsto a_1, \mathsf{ret} \mapsto a_1 \sqcup \mathsf{ret}, t \mapsto t\}$
10	$\{a_1 \mapsto a_1, \mathsf{ret} \mapsto a_1, t \mapsto t\}$
8	$\{a_1 \mapsto a_1, \mathsf{ret} \mapsto \mathsf{ret}, t \mapsto t\}$

$$
\begin{aligned}
[\![(8,\ldots,9)]\!]^\sharp \circ [\![8]\!]^\sharp &= \{a_1 \mapsto a_1, \mathsf{ret} \mapsto a_1, t \mapsto t\} \circ \\
&\quad \{a_1 \mapsto a_1, \mathsf{ret} \mapsto \mathsf{ret}, t \mapsto t\} \\
&= \{a_1 \mapsto a_1, \mathsf{ret} \mapsto a_1, t \mapsto t\}
\end{aligned}
$$

If we know the effects of procedure calls, we can put up a constraint system for determining the abstract state when reaching a program point:

$$
\begin{array}{llll}
\mathcal{R}[\mathsf{main}] & \sqsupseteq & \mathsf{enter}^\sharp\ d_0 & \\
\mathcal{R}[f] & \sqsupseteq & \mathsf{enter}^\sharp\ (\mathcal{R}[u]) & k = (u, f\,();, \_) \quad \text{call} \\
\mathcal{R}[v] & \sqsupseteq & \mathcal{R}[f] & v \quad \text{entry point of} \quad f \\
\mathcal{R}[v] & \sqsupseteq & [\![k]\!]^\sharp\ (\mathcal{R}[u]) & k = (u, \_, v) \quad \text{edge}
\end{array}
$$

... in the Example:



0	$\{a_1 \mapsto \top, \mathsf{ret} \mapsto \top, t \mapsto 0\}$
1	$\{a_1 \mapsto \top, \mathsf{ret} \mapsto \top, t \mapsto 0\}$
2	$\{a_1 \mapsto \top, \mathsf{ret} \mapsto \top, t \mapsto 0\}$
3	$\{a_1 \mapsto \top, \mathsf{ret} \mapsto \top, t \mapsto 0\}$
4	$\{a_1 \mapsto 0, \mathsf{ret} \mapsto \top, t \mapsto 0\}$
5	$\{a_1 \mapsto 0, \mathsf{ret} \mapsto 0, t \mapsto 0\}$
6	$\{a_1 \mapsto 0, \mathsf{ret} \mapsto \top, t \mapsto 0\}$

Discussion:

- At least copy-constants can be determined interprocedurally.
- For that, we had to ignore conditions and complex assignments :-(
- In the second phase, however, we could have been more precise :-)
- The extra abstractions were necessary for two reasons:

  (1) The set of occurring transformers $\mathbb{M} \subseteq \mathbb{D} \to \mathbb{D}$ must be finite;

  (2) The functions $M \in \mathbb{M}$ must be efficiently implementable :-)

- The second condition can, sometimes, be abandoned ...

Observation: Sharir/Pnueli, Cousot

→ Often, procedures are only called for few distinct abstract arguments.

→ Each procedure need only to be analyzed for these :-)

→ Put up a constraint system:

$$
\begin{array}{llll}
[\![v, a]\!]^\sharp & \sqsupseteq & a & v \quad \text{entry point} \\
[\![v, a]\!]^\sharp & \sqsupseteq & \mathsf{combine}^\sharp\,([\![u, a]\!], [\![f, \mathsf{enter}^\sharp\, [\![u, a]\!]^\sharp]\!]^\sharp) & \\
 & & & (u, f();, v) \quad \text{call} \\
[\![v, a]\!]^\sharp & \sqsupseteq & [\![lab]\!]^\sharp\, [\![u, a]\!]^\sharp & k = (u, lab, v) \quad \text{edge} \\
[\![f, a]\!]^\sharp & \sqsupseteq & [\![stop_f, a]\!]^\sharp & stop_f \quad \text{end point of } f
\end{array}
$$

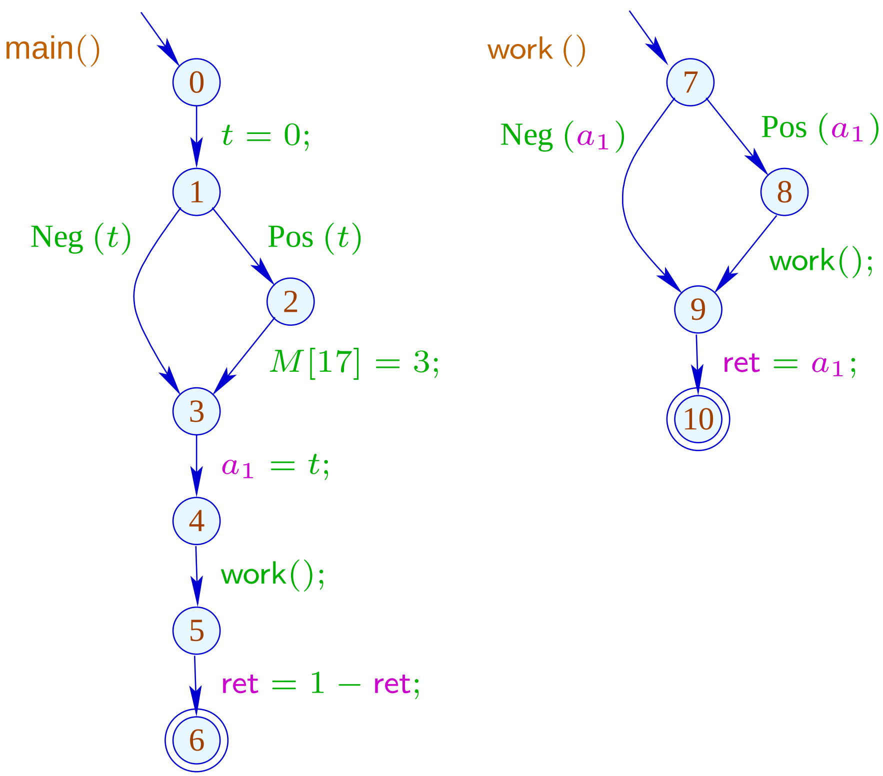
// $[\![v, a]\!]^\sharp$ $==$ value for the argument a .

Discussion:

- This constraint system may be huge :-(
- We do not want to solve it completely!!!
- It is sufficient to compute the correct values for all calls which occur, i.e., which are necessary to determine the value $[\![\mathsf{main}(), a_0]\!]^\sharp$ $\Longrightarrow$ We apply our local fixpoint algorithm :-))
- The fixpoint algo provides us also with the set of actual parameters $a \in \mathbb{D}$ for which procedures are (possibly) called and all abstract values at their program points for each of these calls :-)

... in the Example:

Let us try a full constant propagation ...



	a_1	ret	a_1	ret
0	$\top$	$\top$	$\top$	$\top$
1	$\top$	$\top$	$\top$	$\top$
2	$\top$	$\top$	$\bot$	
3	$\top$	$\top$	$\top$	$\top$
4	$\top$	$\top$	0	$\top$
7	0	$\top$	0	$\top$
8	0	$\top$	$\bot$	
9	0	$\top$	0	$\top$
10	0	$\top$	0	0
5	$\top$	$\top$	0	0
main()	$\top$	$\top$	0	1

Discussion:

- In the Example, the analysis terminates quickly :-)
- If $\mathbb{D}$ has finite height, the analysis terminates if each procedure is only analyzed for finitely many arguments :-))
- Analogous analysis algorithms have proved very effective for the analysis of Prolog :-)
- Together with a points-to analysis and propagation of negative constant information, this algorithm is the heart of a very successful race analyzer for C with Posix threads :-)

(2) The Call-String Approach:

Idea:

→ Compute the set of all reachable call stacks!

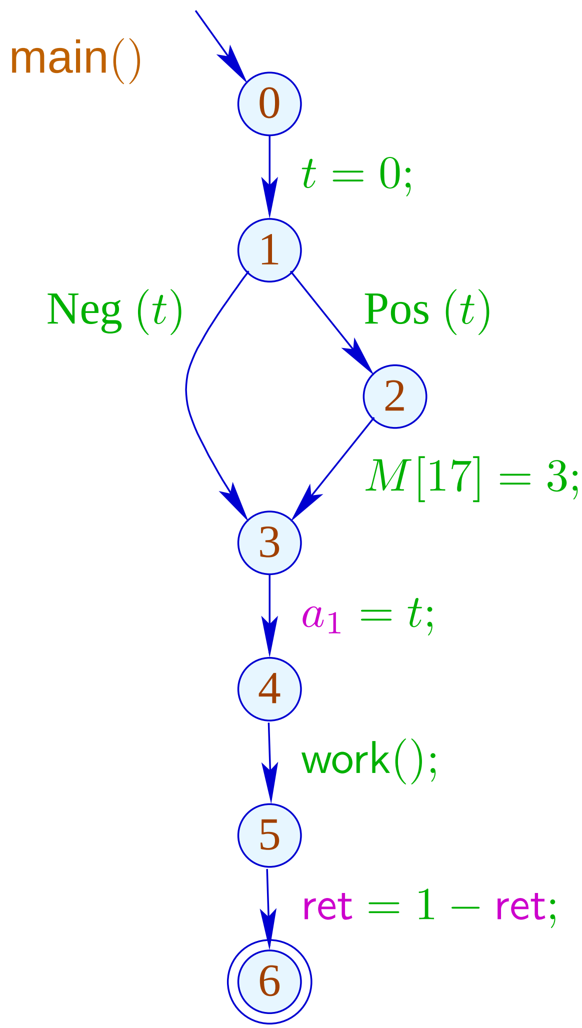
→ In general, this is infinite :-(

→ Only treat stacks up to a fixed depth d precisely! From longer stacks, we only keep the upper prefix of length d :-)

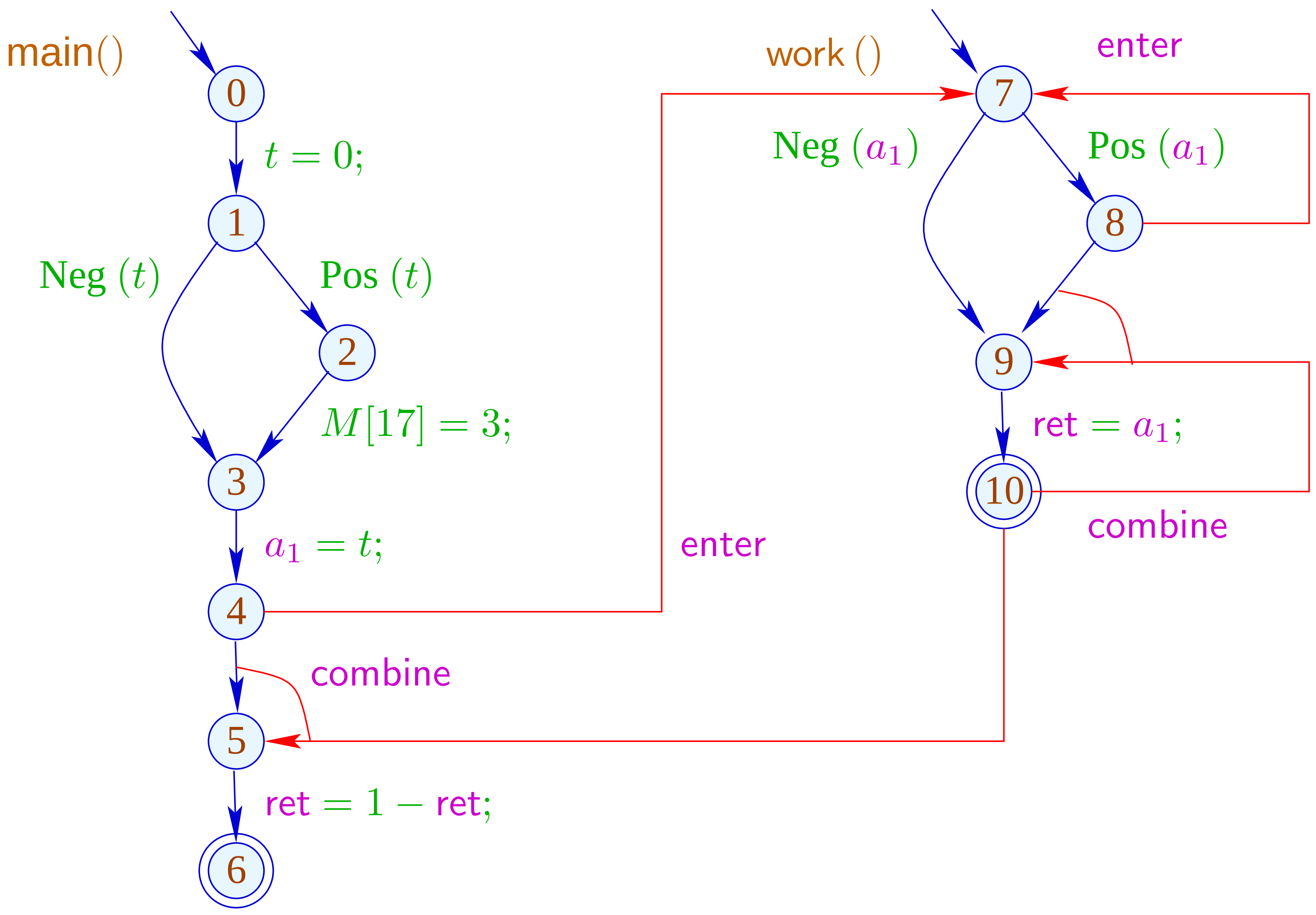
→ Important special case: $d = 0$.

$\Longrightarrow$ Just track the current stack frame ...

... in the Example:



... in the Example:



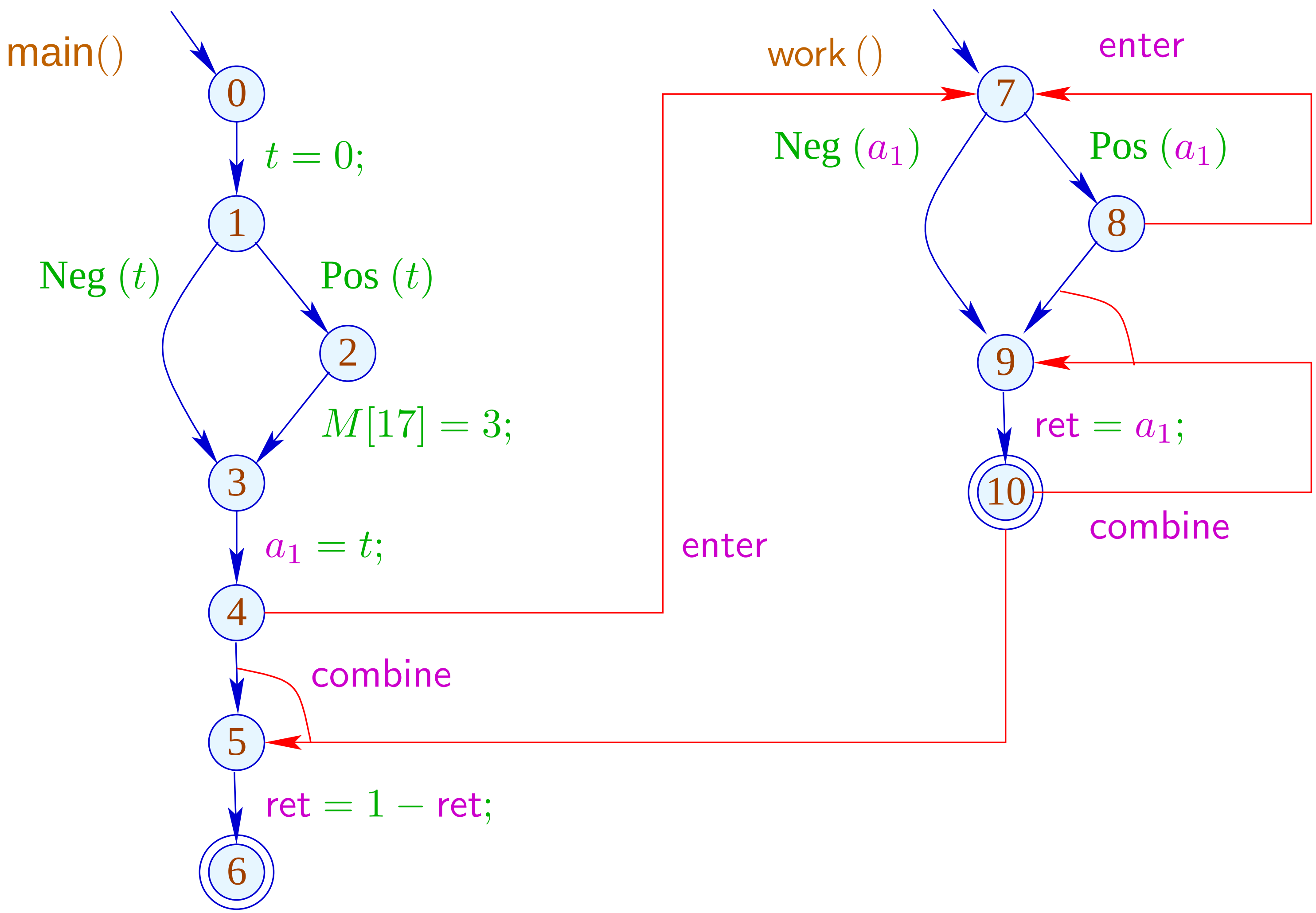
The conditions for $5, 7, 10$, e.g., are:

$$\begin{array}{lll}
\mathcal{R}[5] & \sqsupseteq & \text{combine}^\sharp\,(\mathcal{R}[4], \mathcal{R}[10]) \\[1em]
\mathcal{R}[7] & \sqsupseteq & \text{enter}^\sharp\,(\mathcal{R}[4]) \\
\mathcal{R}[7] & \sqsupseteq & \text{enter}^\sharp\,(\mathcal{R}[8]) \\[1em]
\mathcal{R}[9] & \sqsupseteq & \text{combine}^\sharp\,(\mathcal{R}[8], \mathcal{R}[10])
\end{array}$$

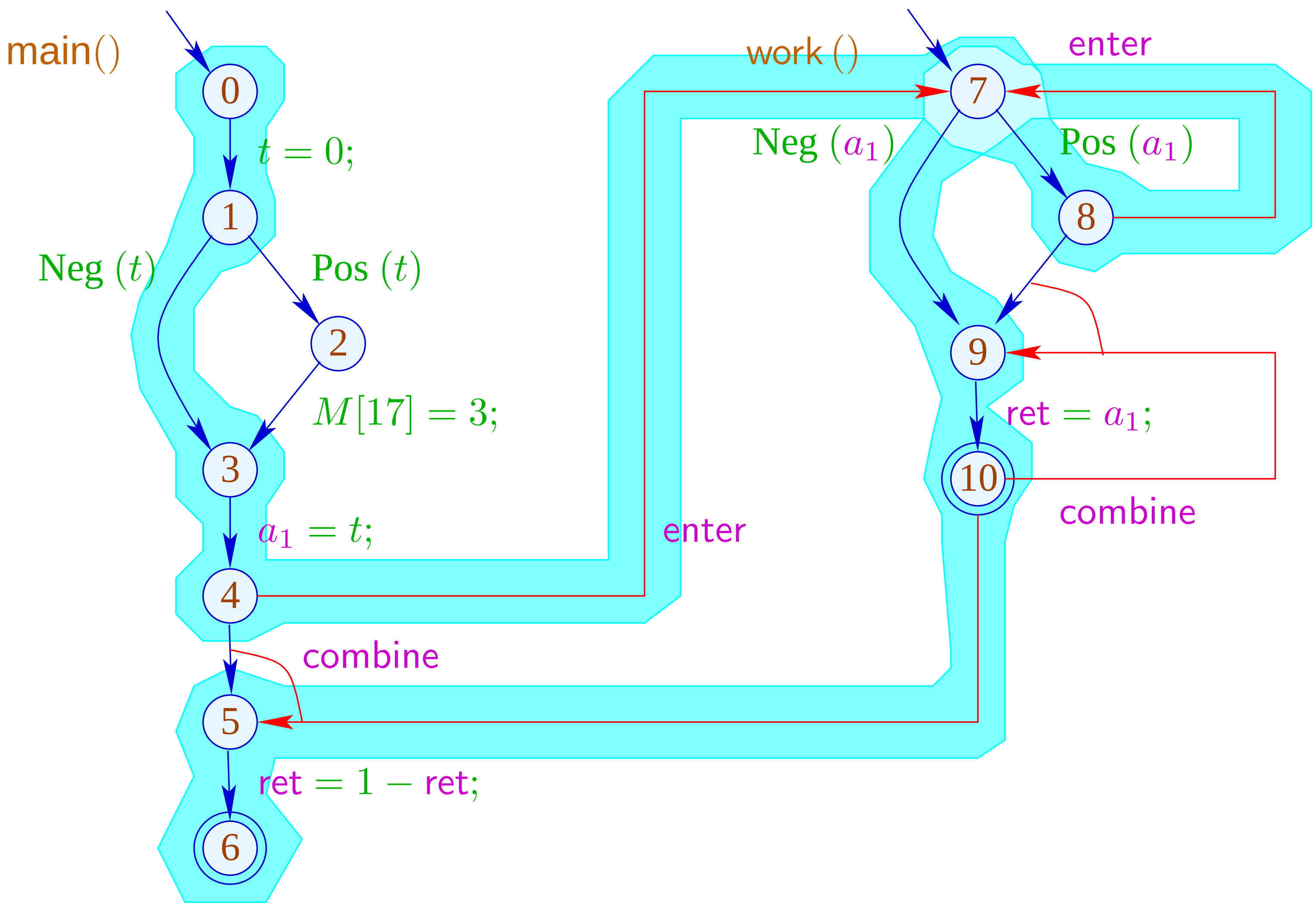
Warning:

The resulting super-graph contains obviously impossible paths ...

... in the Example this is:



... in the Example this is:



Note:

→ In the example, we find the same results:
more paths render the results less precise.

In particular, we provide for each procedure the result just for one (possibly very boring) argument :-(

→ The analysis terminates — whenever $\mathbb{D}$ has no infinite strictly ascending chains :-)

→ The correctness is easily shown w.r.t. the operational semantics with call stacks.

→ For the correctness of the functional approach, the semantics with computation forests is better suited :-)

3 Exploiting Hardware Features

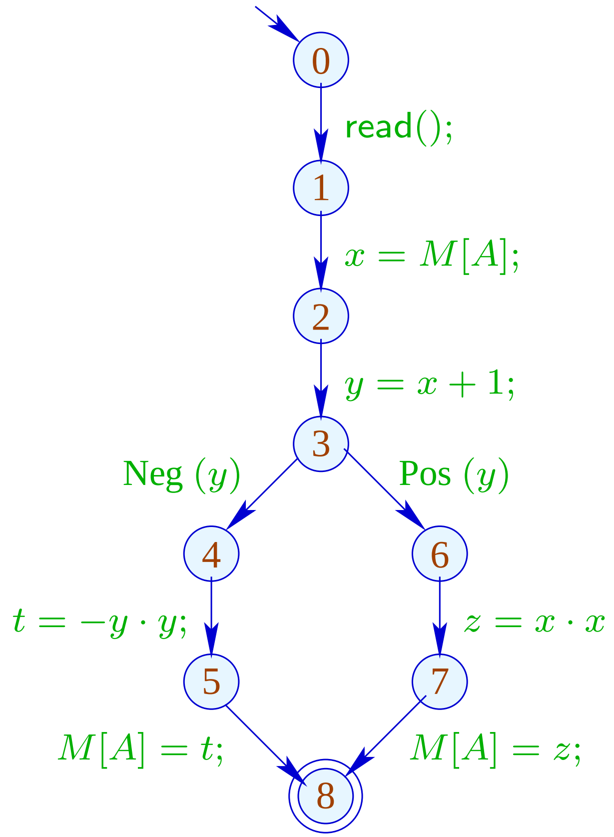
Question: How can we optimally use:

- ... Registers
- ... Pipelines
- ... Caches
- ... Processors ???

3.1 Registers

Example:

```
read();
x = M[A];
y = x + 1;
if (y) {
        z = x · x;
        M[A] = z;
} else {
        t = −y · y;
        M[A] = t;
}
```



The program uses 5 variables ...

Problem:

What if the program uses more variables than there are registers :-(

Idea:

Use one register for several variables :-)

In the example, e.g., one for x, t, z ...

read();

$x = M[A];$

$y = x + 1;$

if (y) $\{$

$\quad z = x \cdot x;$

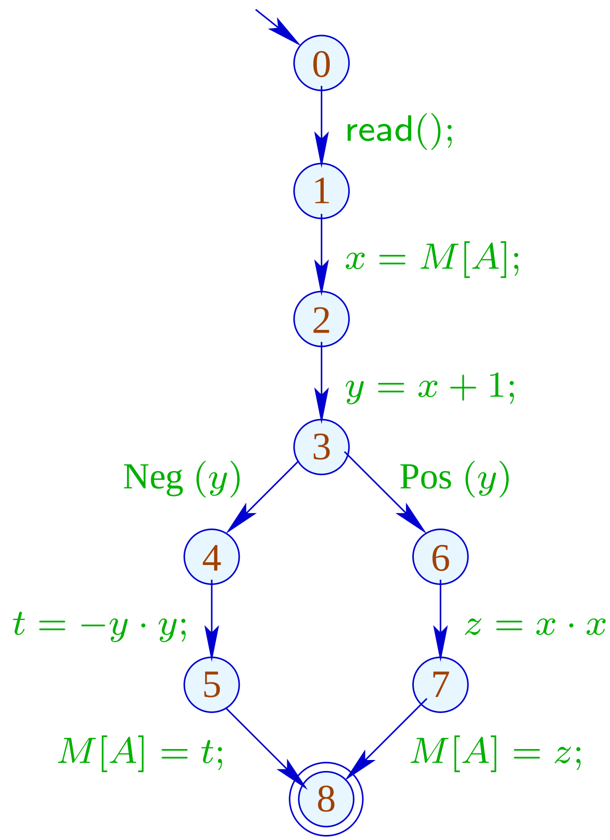
$\quad M[A] = z;$

$\}$ else $\{$

$\quad t = -y \cdot y;$

$\quad M[A] = t;$

$\}$



$\mathsf{read}();$

$R = M[A];$

$y = R + 1;$

$\mathsf{if}\ \ (y)\ \ \{$

$\quad R = R \cdot R;$

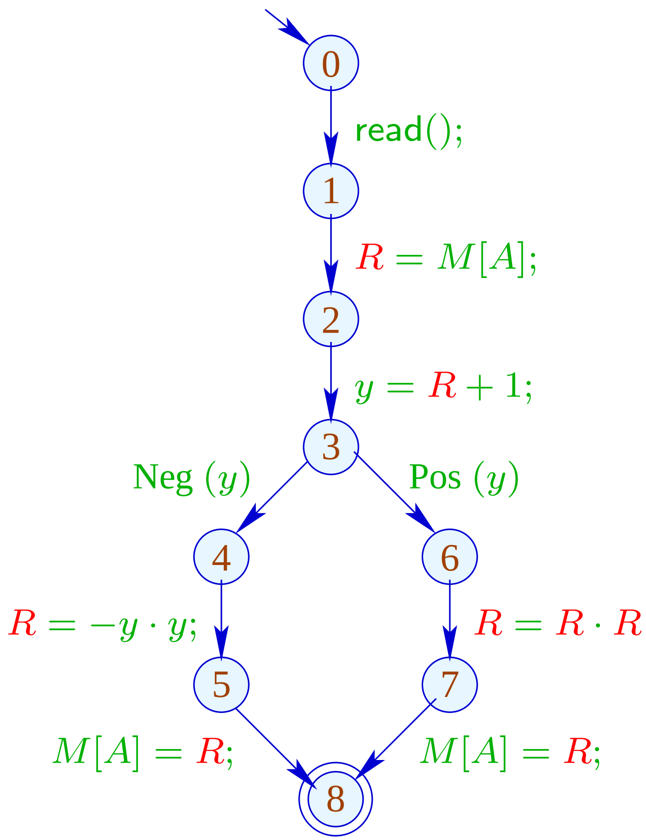
$\quad M[A] = R;$

$\}\ \ \mathsf{else}\ \ \{$

$\quad R = -y \cdot y;$

$\quad M[A] = R;$

$\}$



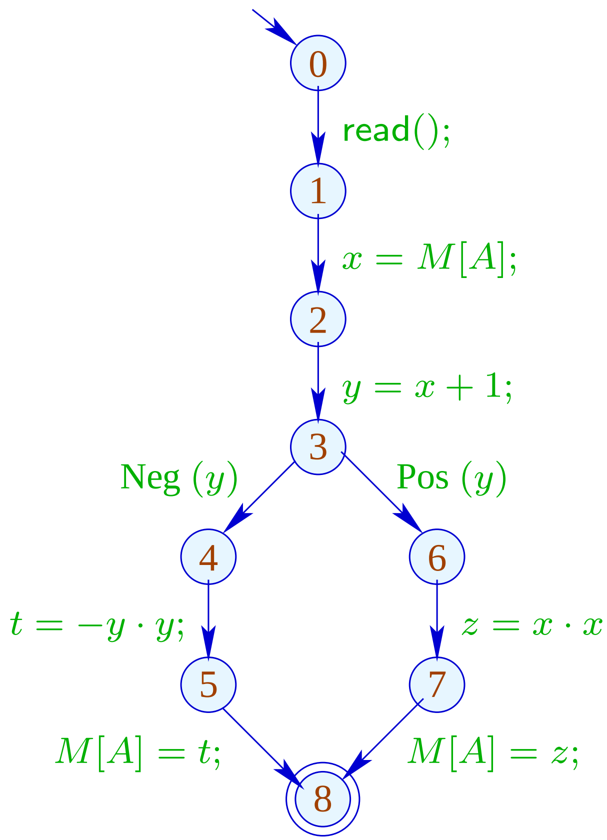
Warning:

This is only possible if the live ranges do not overlap :-)

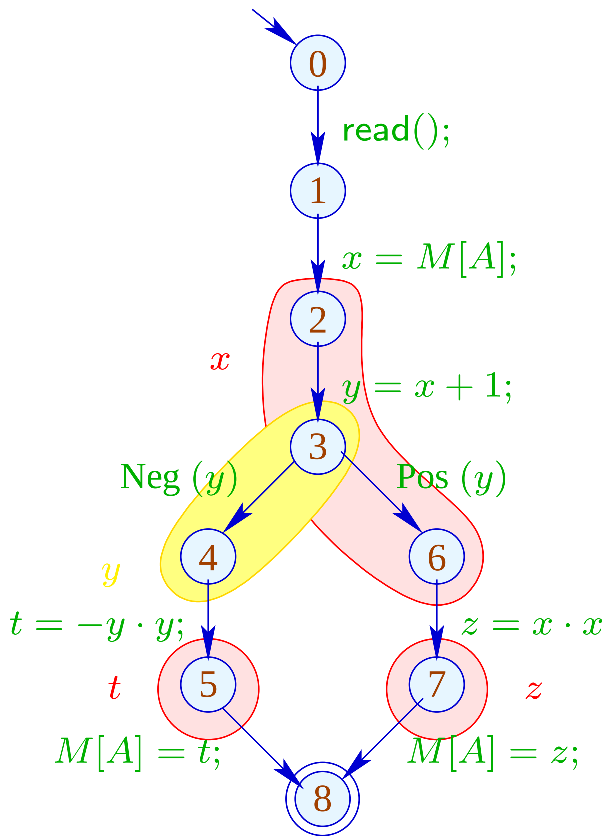
The (true) live range of x is defined by:

$$\mathcal{L}[x] \;=\; \{u \mid x \in \mathcal{L}[u]\}$$

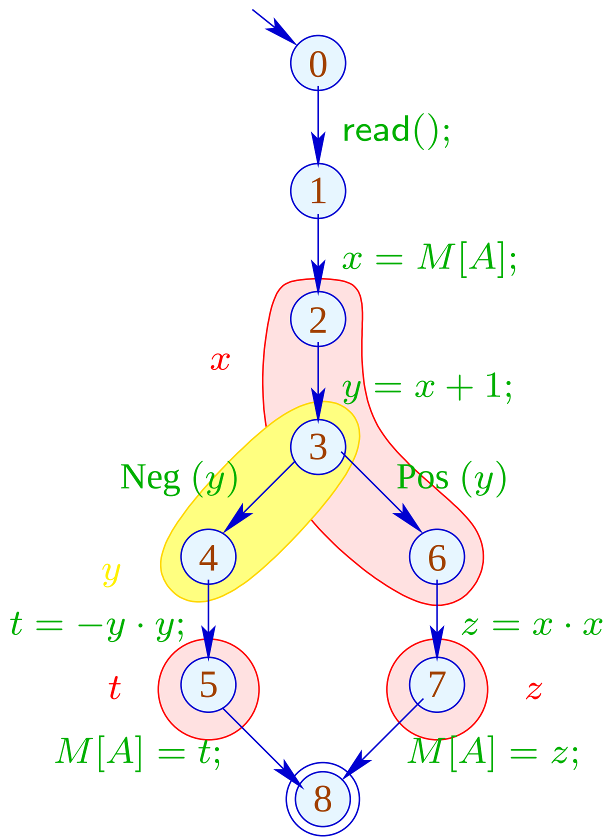
... in the Example:



	$\mathcal{L}$
8	$\emptyset$
7	$\{A, z\}$
6	$\{A, x\}$
5	$\{A, t\}$
4	$\{A, y\}$
3	$\{A, x, y\}$
2	$\{A, x\}$
1	$\{A\}$
0	$\emptyset$



	$\mathcal{L}$
8	$\emptyset$
7	$\{A, z\}$
6	$\{A, x\}$
5	$\{A, t\}$
4	$\{A, y\}$
3	$\{A, x, y\}$
2	$\{A, x\}$
1	$\{A\}$
0	$\{A\}$



Live Ranges:

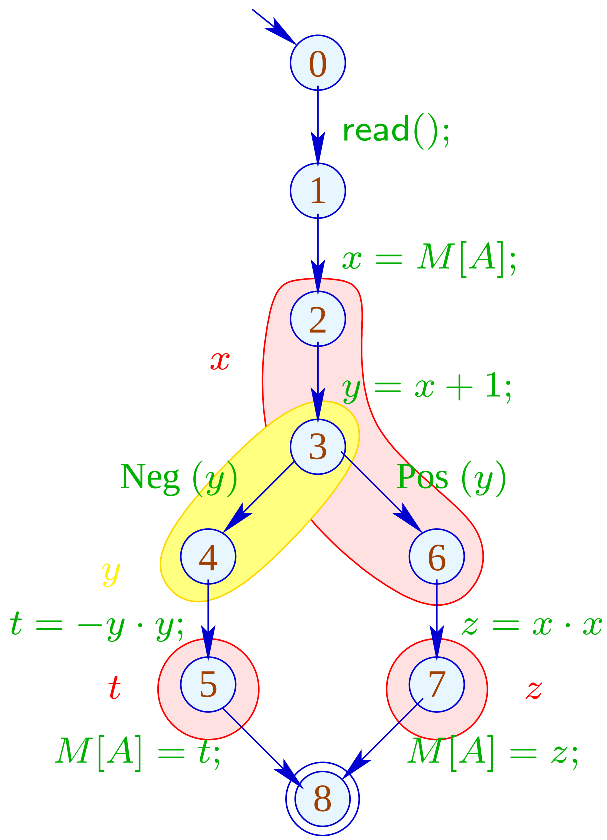
A	$\{0, \ldots, 7\}$
x	$\{2, 3, 6\}$
y	$\{2, 4\}$
t	$\{5\}$
z	$\{7\}$

In order to determine sets of compatible variables, we construct the Interference Graph $I = (Vars, E_I)$ where:

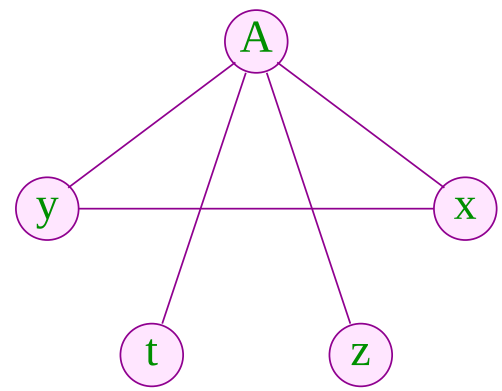
$$E_I = \{\{x, y\} \mid x \neq y, \mathcal{L}[x] \cap \mathcal{L}[y] \neq \emptyset\}$$

E_I has an edge for $x \neq y$ iff x, y are jointly live at some program point :-)

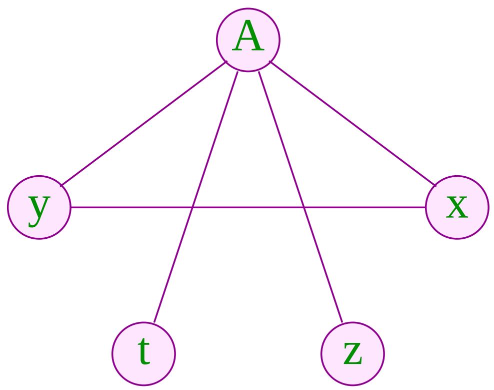
... in the Example:



Interference Graph:



Variables which are not connected with an edge can be assigned to the same register :-)



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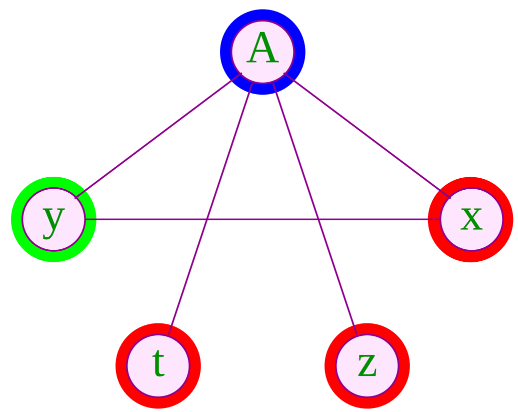

Color $==$ Register

Sviatoslav Sergeevich Lavrov,
Russian Academy of Sciences (1962)

Gregory J. Chaitin, University of Maine (1981)

Abstract Problem:

Given: Undirected Graph (V, E) .

Wanted: Minimal coloring, i.e., mapping $c : V \to \mathbb{N}$ mit

(1) $c(u) \neq c(v)$ for $\{u, v\} \in E$;

(2) $\bigsqcup\{c(u) \mid u \in V\}$ minimal!

- In the example, 3 colors suffice :-) But:
- In general, the minimal coloring is not unique :-(
- It is NP-complete to determine whether there is a coloring with at most k colors :-((

$\Longrightarrow$

We must rely on heuristics or special cases :-)

Greedy Heuristics:

- Start somewhere with color 1;
- Next choose the smallest color which is different from the colors of all already colored neighbors;
- If a node is colored, color all neighbors which not yet have colors;
- Deal with one component after the other ...

... more concretely:

$$
\begin{array}{l}
\textsf{forall} \;\; (v \in V) \;\; c[v] = 0; \\
\textsf{forall} \;\; (v \in V) \;\; \textsf{color}\;(v); \\[1ex]
\textsf{void} \;\; \textsf{color}\;(v) \;\; \{ \\
\qquad \textsf{if} \;\; (c[v] \neq 0) \;\; \textsf{return}; \\
\qquad \textsf{neighbors} = \{u \in V \mid \{u, v\} \in E\}; \\
\qquad c[v] = \bigsqcap\{k > 0 \mid \forall\, u \in \textsf{neighbors} \;:\; k \neq c(u)\}; \\
\qquad \textsf{forall} \;\; (u \in \textsf{neighbors}) \\
\qquad\qquad \textsf{if} \;\; (c(u) == 0) \;\; \textsf{color}\;(u); \\
\}
\end{array}
$$

The new color can be easily determined once the neighbors are sorted according to their colors :-)

Discussion:

→ Essentially, this is a Pre-order DFS :-)

→ In theory, the result may arbitrarily far from the optimum :-(

→ ... in practice, it may not be as bad :-)

→ ... Anecdote: different variants have been patented !!!

Discussion:

→ Essentially, this is a Pre-order DFS :-)

→ In theory, the result may arbitrarily far from the optimum :-(

→ ... in practice, it may not be as bad :-)

→ ... Anecdote: different variants have been patented !!!

The algorithm works the better the smaller life ranges are ...

Idea: Life Range Splitting