$$\begin{aligned}
[\![R = M[e];]\!]\,(\rho,\mu) &= (\boxed{\rho \oplus \{R \mapsto \mu([\![e]\!]\,\rho))\}}\,, \mu) \\
[\![M[e_1] = e_2;]\!]\,(\rho,\mu) &= (\rho, \boxed{\mu \oplus \{[\![e_1]\!]\,\rho \mapsto [\![e_2]\!]\,\rho\}}\,)
\end{aligned}$$

Example:

$[\![x = x + 1;]\!]\,(\{x \mapsto 5\}, \mu) = (\rho, \mu)$ **where:**

$$\begin{aligned}
\rho &= \{x \mapsto 5\} \oplus \{x \mapsto [\![x + 1]\!]\,\{x \mapsto 5\}\} \\
&= \{x \mapsto 5\} \oplus \{x \mapsto 6\} \\
&= \{x \mapsto 6\}
\end{aligned}$$

A path $\pi = k_1 k_2 \ldots k_m$ is a computation for the state s if:

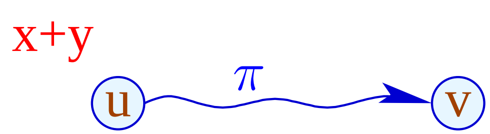
$$s \in \mathit{def}\ (\llbracket k_m \rrbracket \circ \ldots \circ \llbracket k_1 \rrbracket)$$

The result of the computation is:

$$\llbracket \pi \rrbracket\, s = (\llbracket k_m \rrbracket \circ \ldots \circ \llbracket k_1 \rrbracket)\, s$$

Application:

Assume that we have computed the value of $x + y$ at program point u:



We perform a computation along path π and reach v where we evaluate again $x + y$ ...

Idea:

If x and y have not been modified in π, then evaluation of $x + y$ at v must return the same value as evaluation at u **:-)**

We can check this property at every edge in π **:-}**

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More generally:

Assume that the values of the expressions $A = \{e_1, \ldots, e_r\}$ are available at u.

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More generally:

Assume that the values of the expressions $A = \{e_1, \ldots, e_r\}$ are available at u.

Every edge k transforms this set into a set $[\![k]\!]^\sharp\, A$ of expressions whose values are available after execution of k ...

... which transformations can be composed to the effect of a path $\pi = k_1 \dots k_r$:

$$[\![\pi]\!]^\sharp = [\![k_r]\!]^\sharp \circ \dots \circ [\![k_1]\!]^\sharp$$

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The effect $[\![k]\!]^\sharp$ of an edge $k = (u, lab, v)$ only depends on the label lab, i.e., $[\![k]\!]^\sharp = [\![lab]\!]^\sharp$

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The effect $[\![k]\!]^\sharp$ of an edge $k = (u, lab, v)$ only depends on the label lab, i.e., $[\![k]\!]^\sharp = [\![lab]\!]^\sharp$ where:

$$\begin{array}{lcl} [\![;]\!]^\sharp\, A & = & A \\ [\![Pos(e)]\!]^\sharp\, A & = & [\![Neg(e)]\!]^\sharp\, A \quad = \quad A \cup \{e\} \\ [\![x = e;]\!]^\sharp\, A & = & (A \cup \{e\}) \backslash Expr_x \qquad \text{where} \\ & & Expr_x \text{ all expressions which contain } x \end{array}$$

$$
\begin{aligned}
[\![x = M[e];]\!]^\sharp\, A &= (A \cup \{e\}) \backslash Expr_x \\
[\![M[e_1] = e_2;]\!]^\sharp\, A &= A \cup \{e_1, e_2\}
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By that, every path can be analyzed :-)

A given program may admit several paths :-(

For any given input, another path may be chosen :-((

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$\Longrightarrow$ We require the set:

$$
\mathcal{A}[v] \quad = \quad \bigcap \{[\![\pi]\!]^\sharp \emptyset \mid \pi : start \to^* v\}
$$

Concretely:

- $\rightarrow$ We consider all paths π which reach v.
- $\rightarrow$ For every path π, we determine the set of expressions which are available along π.
- $\rightarrow$ Initially at program start, nothing is available :-)
- $\rightarrow$ We compute the intersection $\Longrightarrow$ safe information

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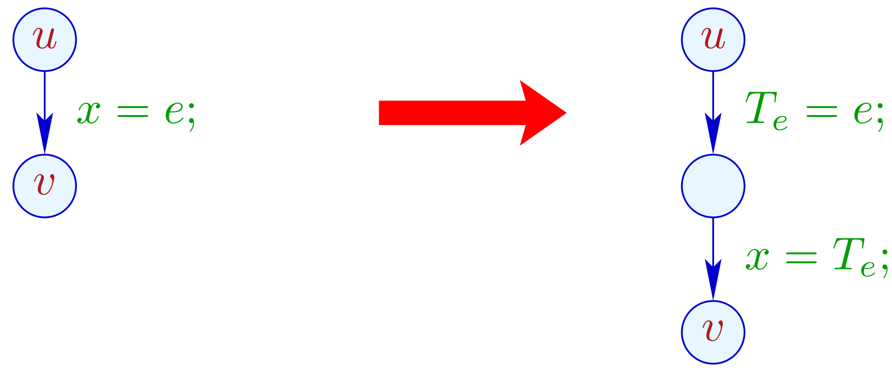
How do we exploit this information ???

Transformation 1.1:

We provide novel registers T_e as storage for the e:



Transformation 1.1:

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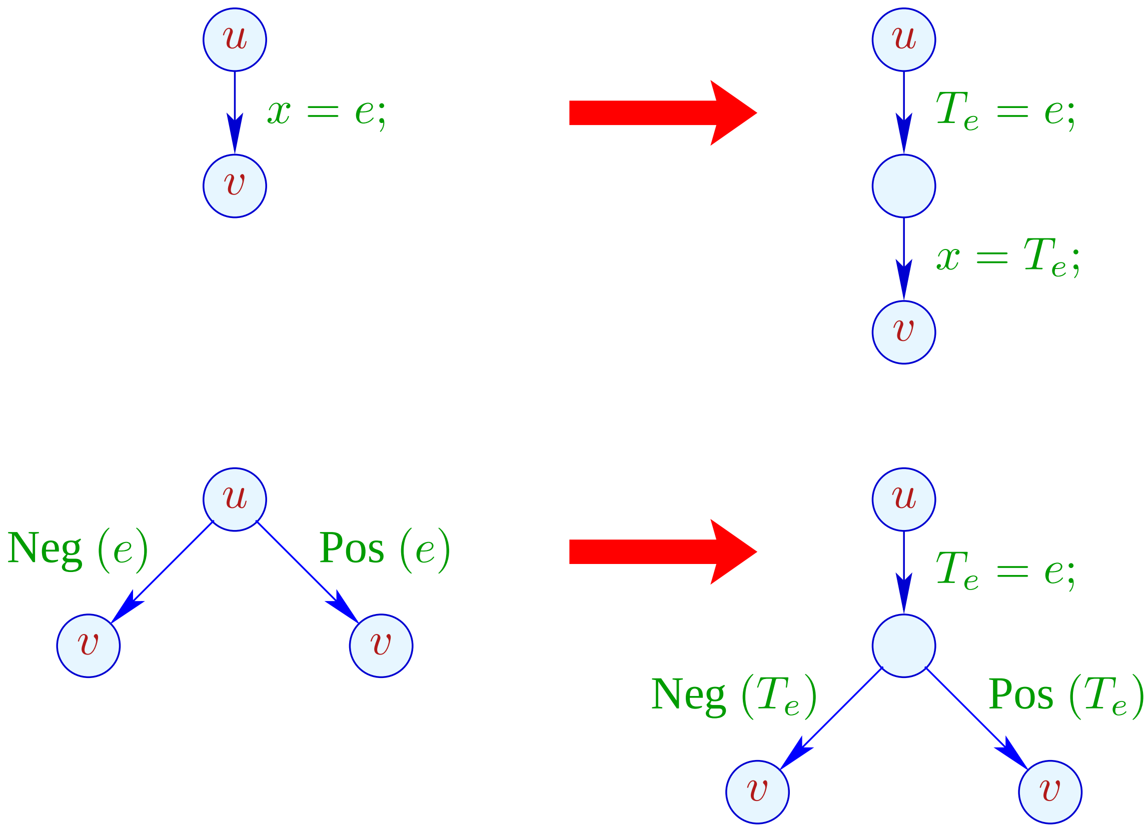
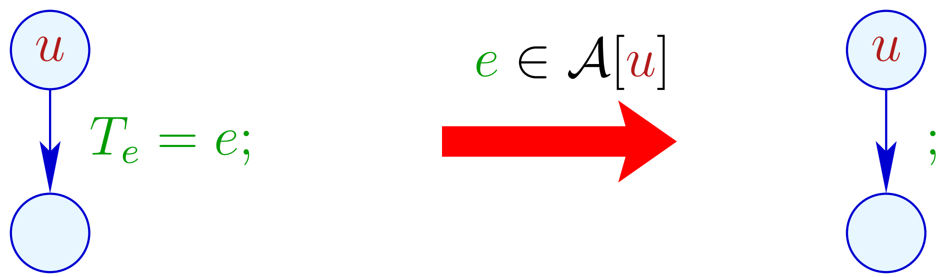


... analogously for $R = M[e];$ and $M[e_1] = e_2;$.

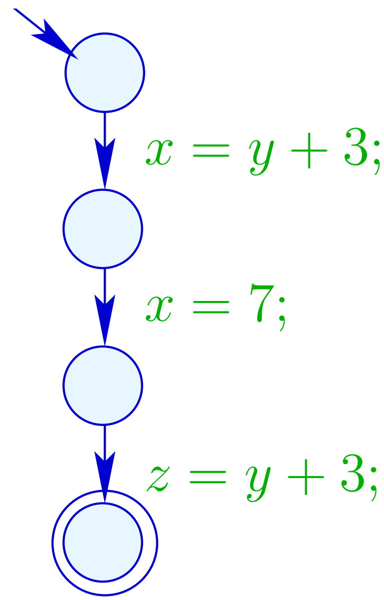
Transformation 1.2:

If e is available at program point u, then e need not be re-evaluated:



We replace the assignment with Nop :-)

Example:



$$
\begin{aligned}
x &= y + 3; \\
x &= 7; \\
z &= y + 3;
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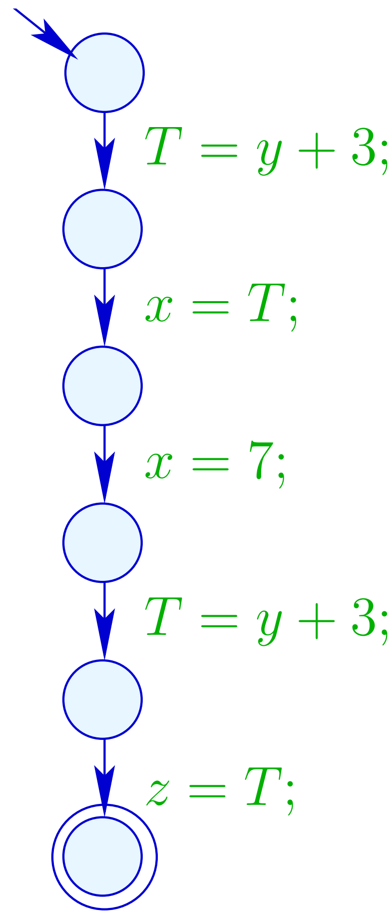

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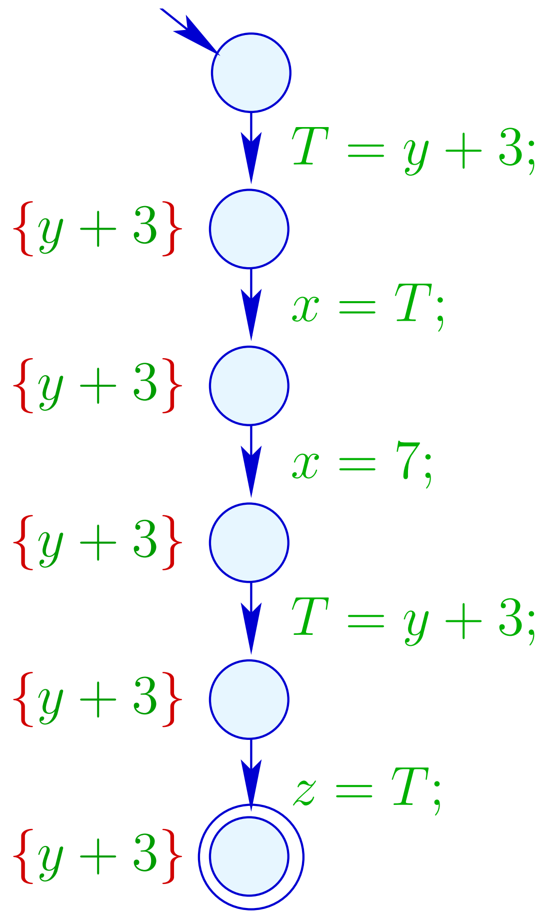

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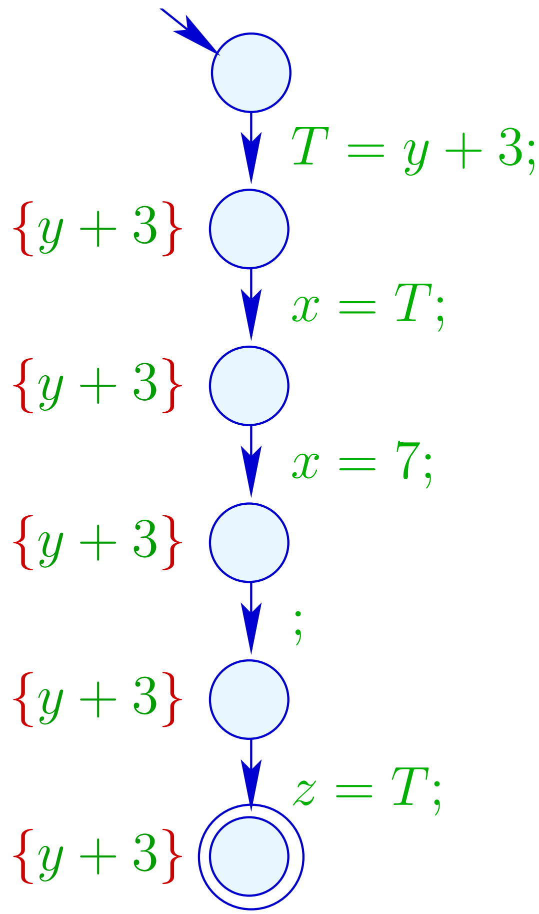


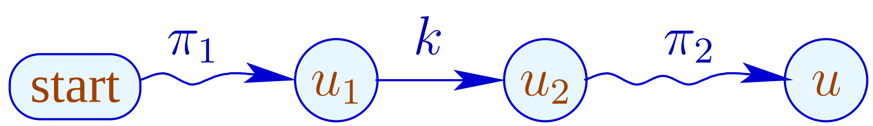
Correctness: (Idea)

Transformation 1.1 preserves the semantics and $\mathcal{A}[u]$ for all program points u :-)

Assume $\pi : start \to^* u$ is the path taken by a computation.

If $e \in \mathcal{A}[u]$, then also $e \in [\![\pi]\!]^\sharp\,\emptyset$.

Therefore, π can be decomposed into:



with the following properties:

- The expression e is evaluated at the edge k;
- The expression e is not removed from the set of available expressions at any edge in π_2, i.e., no variable of e receives a new value :-)

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$$\Longrightarrow$$

The register T_e contains the value of e whenever u is reached :-))

Warning:

Transformation 1.1 is only meaningful for assignments $x = e;$ where:

$\rightarrow$ $e \notin Vars$;

$\rightarrow$ the evaluation of e is non-trivial :-}

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$\rightarrow$ $x \notin Vars(e)$;

$\rightarrow$ $e \notin Vars$;

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Which leaves us with the following question ...

Question:

How do we compute $\mathcal{A}[u]$ for every program point u ??

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How can we compute $\mathcal{A}[u]$ for every program point u ??

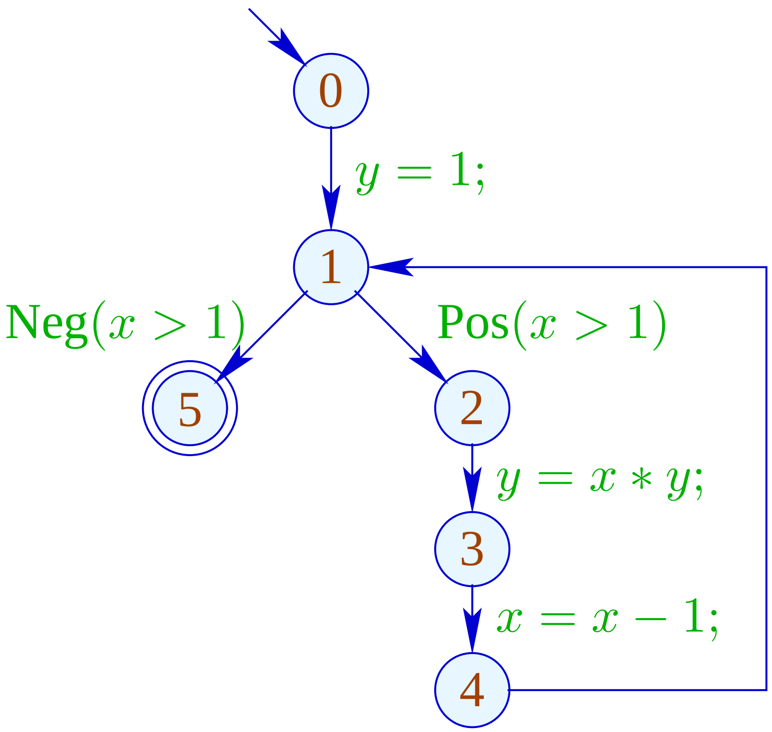
We collect all restrictions to the values of $\mathcal{A}[u]$ into a system of constraints:

$$\begin{array}{lll} \mathcal{A}[start] & \subseteq & \emptyset \\ \mathcal{A}[v] & \subseteq & [\![k]\!]^\sharp(\mathcal{A}[u]) \qquad\qquad k = (u, \_, v) \quad \text{edge} \end{array}$$

Wanted:

- a maximally large solution (??)
- an algorithm which computes this :-)

Example:



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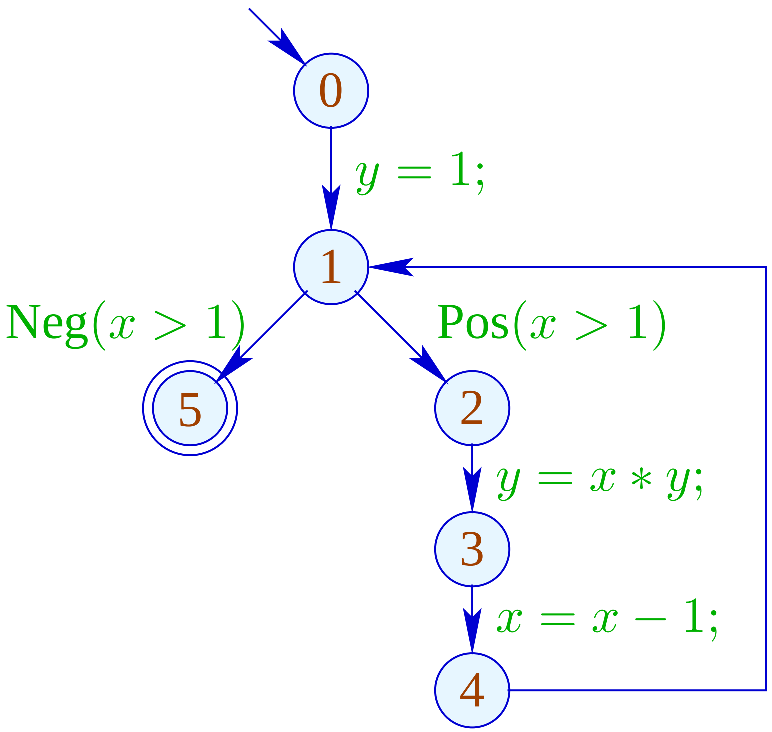

$$\mathcal{A}[0] \quad \subseteq \quad \emptyset$$

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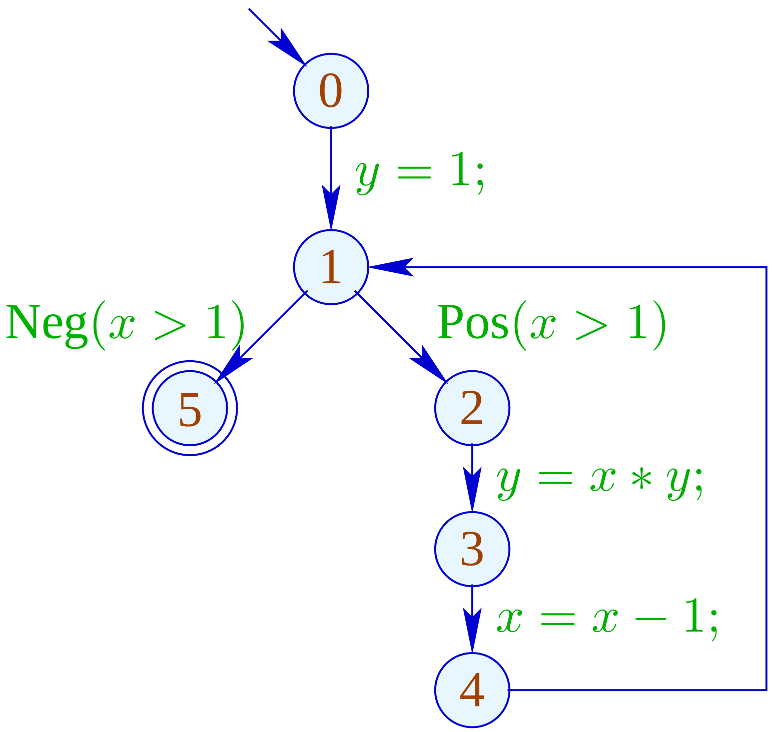

$$\begin{array}{lll} \mathcal{A}[0] & \subseteq & \emptyset \\ \mathcal{A}[1] & \subseteq & (\mathcal{A}[0] \cup \{1\}) \backslash Expr_y \\ \mathcal{A}[1] & \subseteq & \mathcal{A}[4] \end{array}$$

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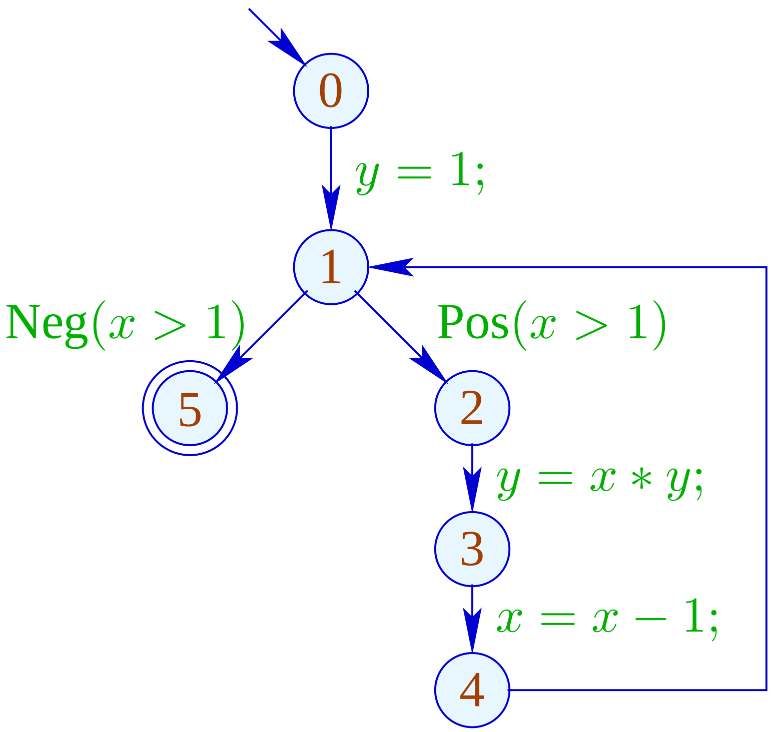

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\end{array}
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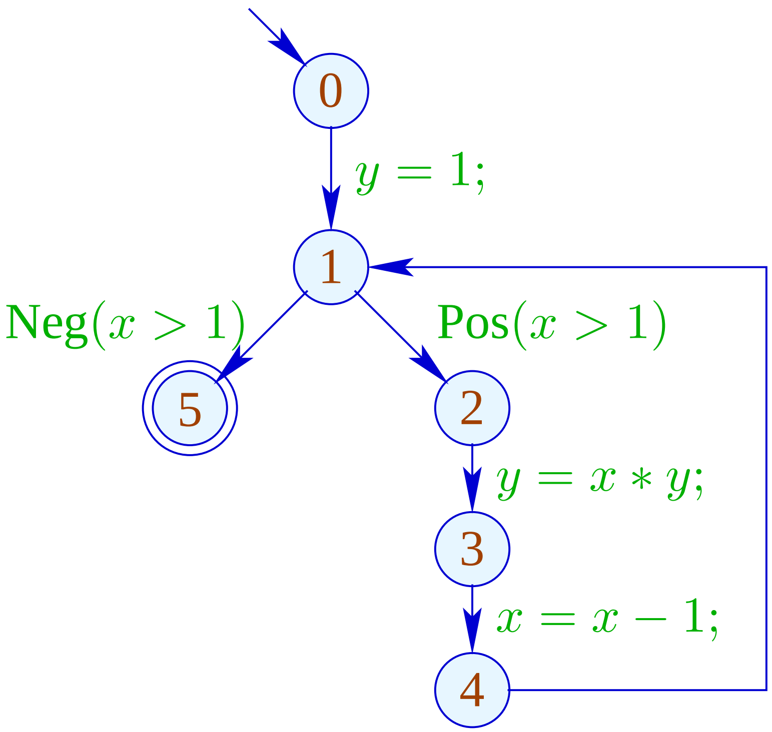

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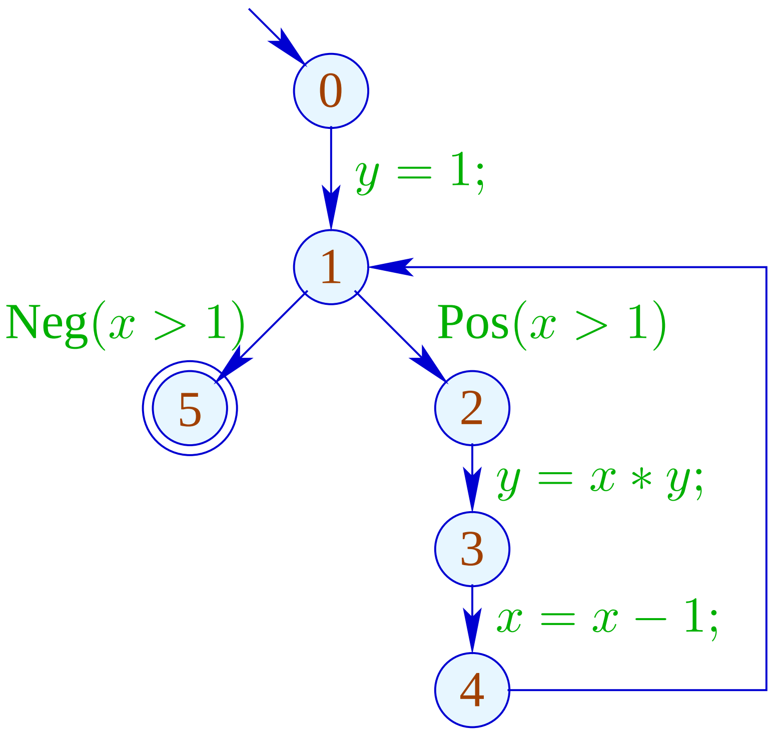

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$$

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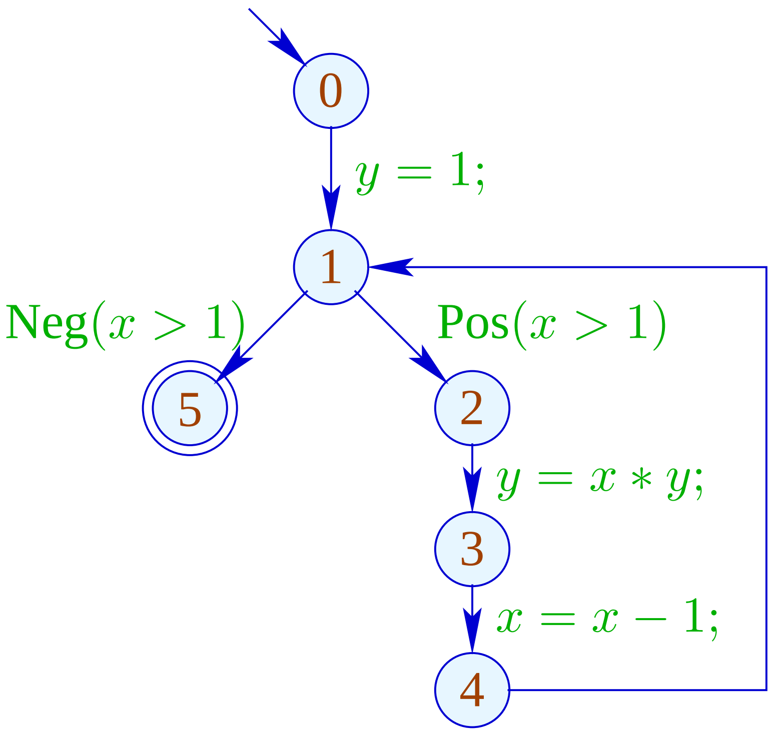
Example:



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\end{array}
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Wanted:

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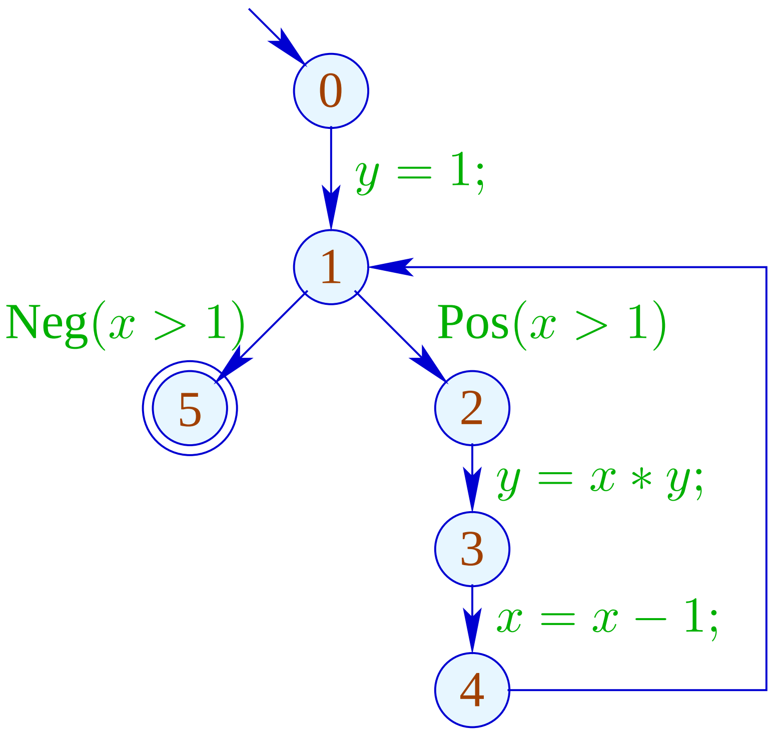
Example:



Solution:

$$
\begin{aligned}
\mathcal{A}[0] &= \emptyset \\
\mathcal{A}[1] &= \{1\} \\
\mathcal{A}[2] &= \{1, x > 1\} \\
\mathcal{A}[3] &= \{1, x > 1\} \\
\mathcal{A}[4] &= \{1\} \\
\mathcal{A}[5] &= \{1, x > 1\}
\end{aligned}
$$

Observation:

- The possible values for $\mathcal{A}[u]$ form a complete lattice:

$$\mathbb{D} = 2^{Expr} \quad \text{with} \quad B_1 \sqsubseteq B_2 \quad \text{iff} \quad B_1 \supseteq B_2$$

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$$\mathbb{D} = 2^{Expr} \quad \text{with} \quad B_1 \sqsubseteq B_2 \quad \text{iff} \quad B_1 \supseteq B_2$$

- The functions $[\![k]\!]^\sharp : \mathbb{D} \to \mathbb{D}$ are monotonic, i.e.,

$$[\![k]\!]^\sharp(B_1) \sqsubseteq [\![k]\!]^\sharp(B_2) \quad \text{whenever} \quad B_1 \sqsubseteq B_2$$