Example: $\{x \mapsto 1, y \mapsto -7\} \; \Delta \; \{x \mapsto \top, y \mapsto -7\}$

(3) States:

$$\Delta \;\subseteq\; ((\mathit{Vars} \to \mathbb{Z}) \times (\mathbb{N} \to \mathbb{Z})) \times (\mathit{Vars} \to \mathbb{Z}^{\top})_{\bot}$$

$$(\rho, \mu) \;\Delta\; D \qquad \text{iff} \qquad \rho \;\Delta\; D$$

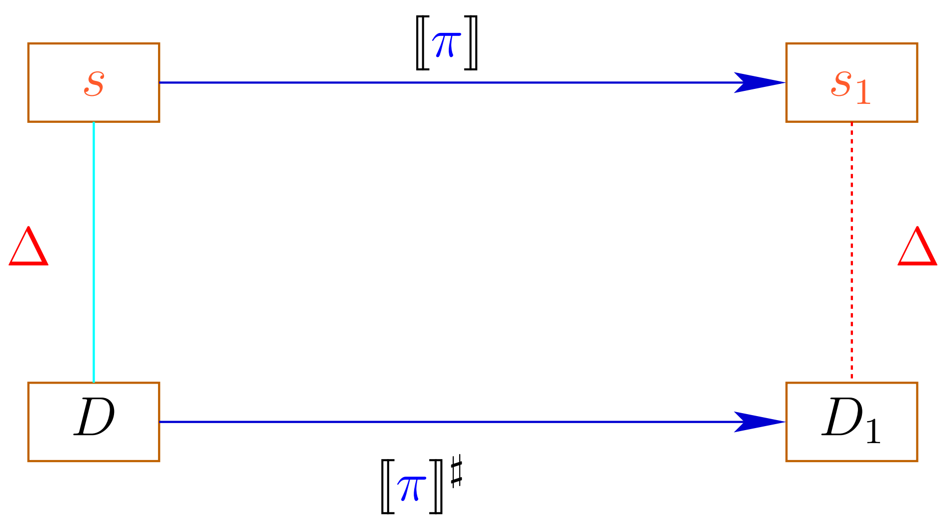
Concretization:

$$\gamma\, D = \begin{cases} \emptyset & \text{if} \quad D = \bot \\ \{(\rho, \mu) \mid \forall\, x : \; (\rho\, x) \;\Delta\; (D\, x)\} & \text{otherwise} \end{cases}$$

We show:

$(*)$ If $s \,\Delta\, D$ and $[\![\pi]\!]\, s$ is defined, then:

$$([\![\pi]\!]\, s) \;\; \Delta \;\; ([\![\pi]\!]^\sharp\, D)$$



The abstract semantics simulates the concrete semantics :-)

In particular:

$$[\![\pi]\!] \, s \in \gamma \left([\![\pi]\!]^\sharp \, D \right)$$

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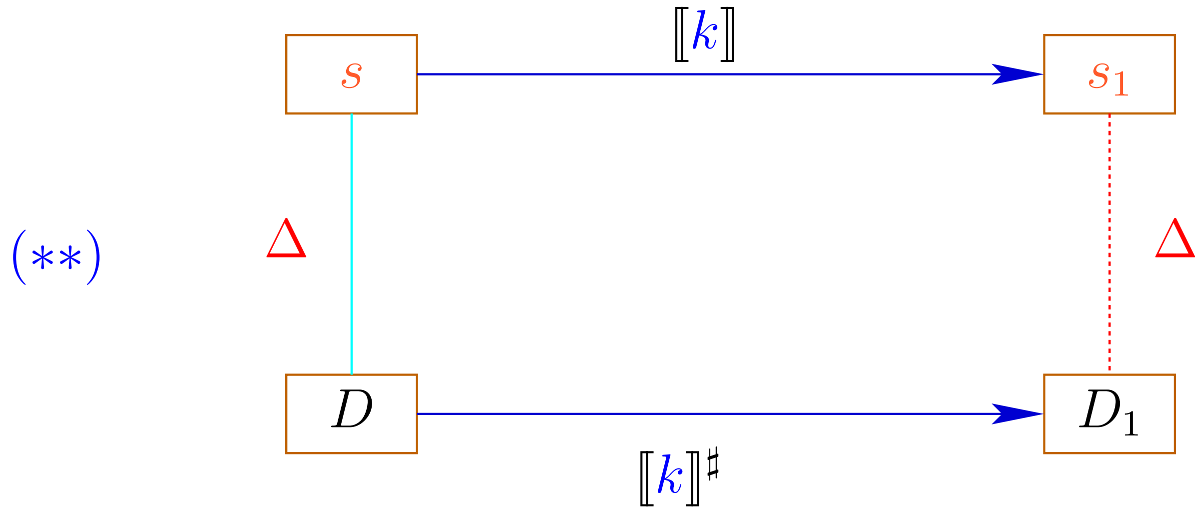
$$[\![\pi]\!]\, s \in \gamma\, ([\![\pi]\!]^\sharp\, D)$$

In practice, this means, e.g., that $D\, x = -7$ implies:

$$\begin{array}{rrcll} & \rho'\, x & = & -7 & \text{for all} \quad \rho' \in \gamma\, D \\ \Longrightarrow & \rho_1\, x & = & -7 & \text{for} \quad (\rho_1, \_) = [\![\pi]\!]\, s \end{array}$$

To prove $(*)$, we show for every edge k :



Then $(*)$ follows by induction **:-)**

To prove $(**)$, we show for every expression e :

$$(***) \quad ([\![e]\!]\,\rho) \;\Delta\; ([\![e]\!]^\sharp\, D) \quad \text{whenever} \quad \rho\,\Delta\,D$$

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$$(x \,\Box\, y) \;\Delta\; (x^\sharp \,\Box^\sharp\, y^\sharp) \qquad \text{whenever} \quad x \,\Delta\, x^\sharp \wedge y \,\Delta\, y^\sharp$$

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This precisely was how we have defined the operators $\Box^\sharp$:-)

Now, $(**)$ is proved by case distinction on the edge labels *lab* .

Let $s = (\rho, \mu) \;\Delta\; D$. In particular, $\bot \neq D \;:\; Vars \to \mathbb{Z}^\top$

Case $\boxed{x = e;}$:

$$\rho_1 = \rho \oplus \{x \mapsto [\![e]\!]\,\rho\} \qquad \mu_1 = \mu$$
$$D_1 = D \oplus \{x \mapsto [\![e]\!]^\sharp\, D\}$$

$$\Longrightarrow \qquad (\rho_1, \mu_1) \;\Delta\; D_1$$

Case $\boxed{x = M[e];}$:

$$\begin{array}{lcl} \rho_1 & = & \rho \oplus \{x \mapsto \mu\,(\llbracket e \rrbracket^\sharp \rho)\} \\ D_1 & = & D \oplus \{x \mapsto \top\} \end{array} \qquad \mu_1 = \mu$$

$$\Longrightarrow \qquad (\rho_1, \mu_1) \;\Delta\; D_1$$

Case $\boxed{M[e_1] = e_2;}$:

$$\begin{array}{lcl} \rho_1 & = & \rho \\ D_1 & = & D \end{array} \qquad \mu_1 = \mu \oplus \{\llbracket e_1 \rrbracket^\sharp \rho \mapsto \llbracket e_2 \rrbracket^\sharp \rho\}$$

$$\Longrightarrow \qquad (\rho_1, \mu_1) \;\Delta\; D_1$$

Case $\boxed{\mathrm{Neg}(e)}$: $(\rho_1, \mu_1) = s$ where:

$$\begin{array}{rccl} & 0 & = & [\![e]\!]\,\rho \\ & & \Delta & [\![e]\!]^\sharp\, D \\ \Longrightarrow & 0 & \sqsubseteq & [\![e]\!]^\sharp\, D \\ \Longrightarrow & \bot & \neq & D_1 \;=\; D \\ \Longrightarrow & \multicolumn{3}{l}{(\rho_1, \mu_1)\ \Delta\ D_1} \end{array}$$

Case $\boxed{\text{Pos}(e)}$: $(\rho_1, \mu_1) = s$ where:

$$
\begin{array}{rccl}
& 0 & \neq & [\![e]\!]\,\rho \\
& & \Delta & [\![e]\!]^\sharp\, D \\
\Longrightarrow & 0 & \neq & [\![e]\!]^\sharp\, D \\
\Longrightarrow & \bot & \neq & D_1 \;=\; D \\
\Longrightarrow & (\rho_1, \mu_1) \;\Delta\; D_1 & &
\end{array}
$$

:-)

We conclude: The assertion $(*)$ is true :-))

The MOP-Solution:

$$\mathcal{D}^*[v] = \bigsqcup \{ [\![\pi]\!]^\sharp \, D_\top \mid \pi : start \to^* v \}$$

where $D_\top \, x = \top \qquad (x \in Vars)$.

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By $(*)$, we have for all initial states s and all program executions π which reach v :

$$([\![\pi]\!]\, s) \;\Delta\; (\mathcal{D}^*[v])$$

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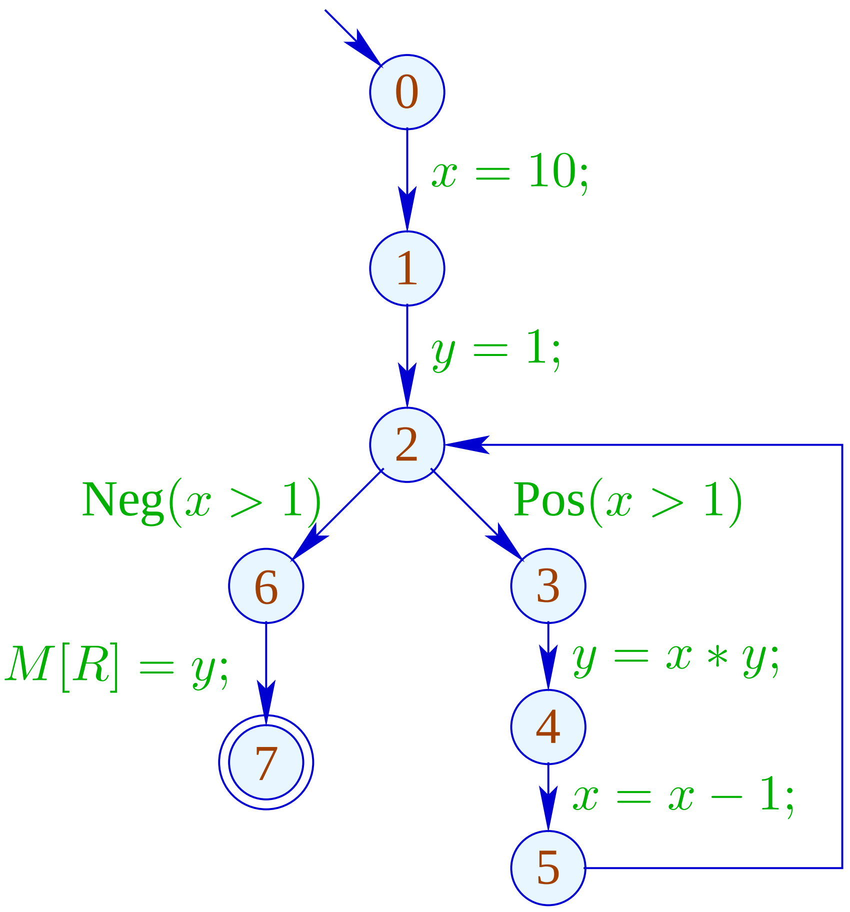
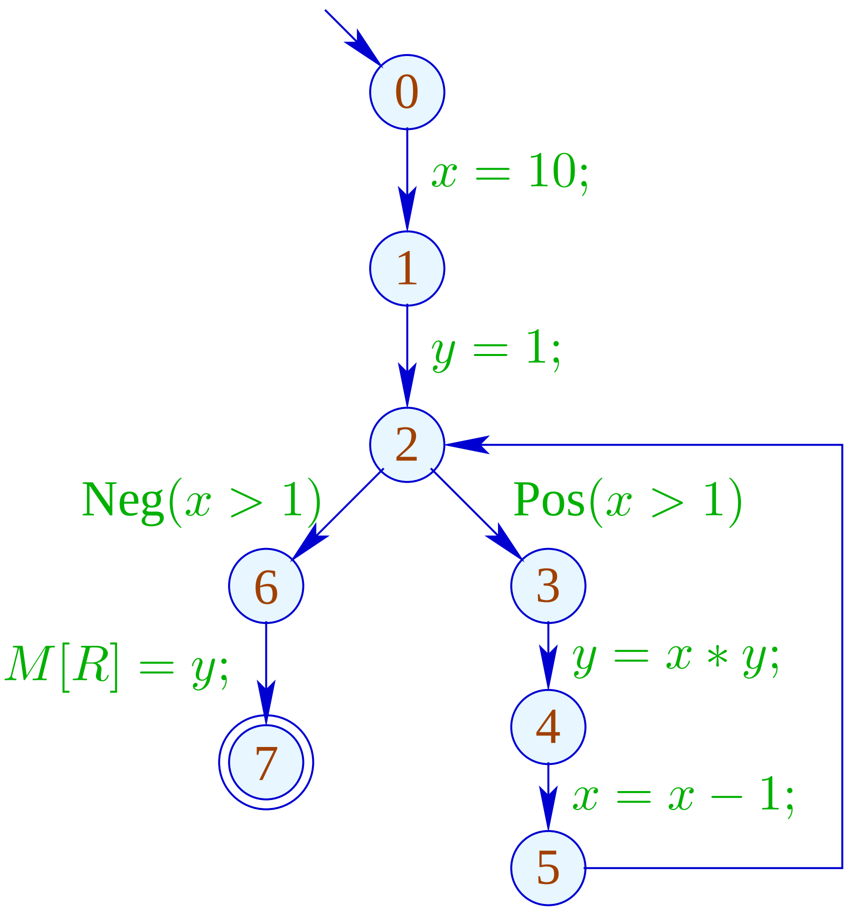
In order to approximate the MOP, we use our constraint system :-))

Example:

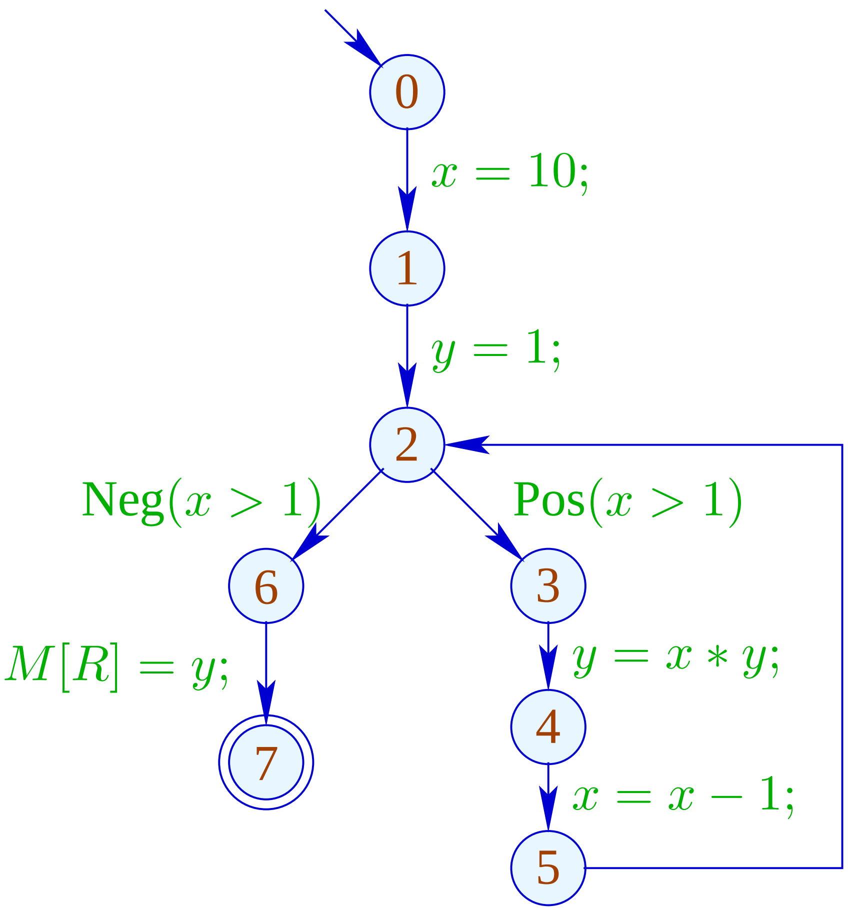


Example:



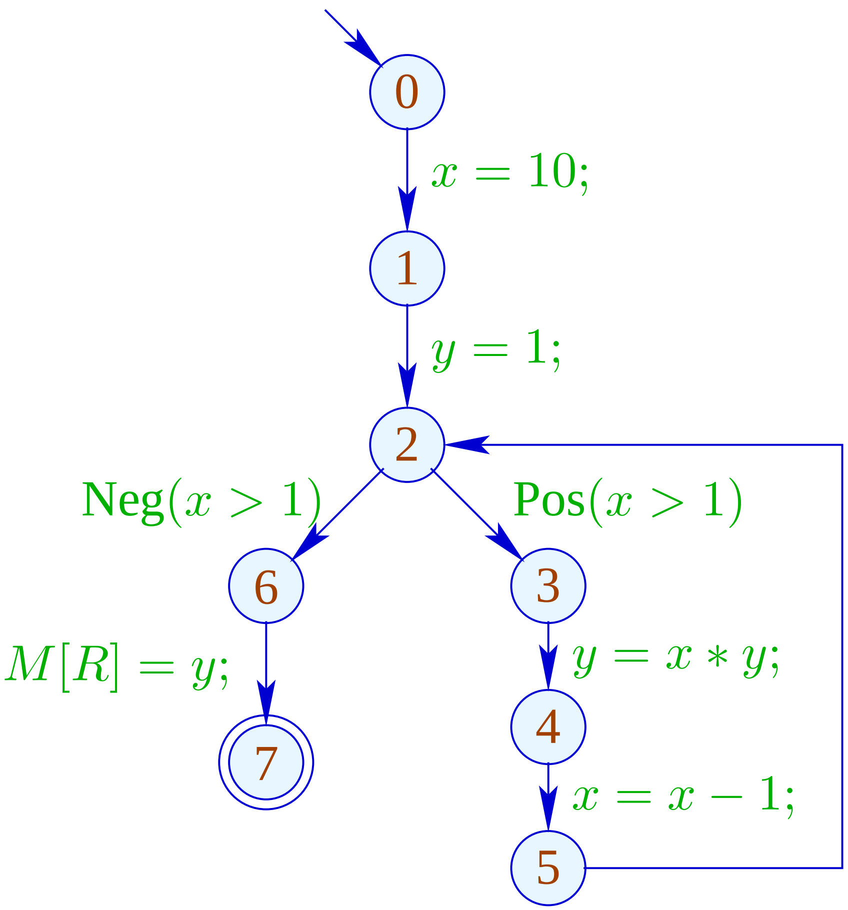
	1	
	x	y
0	$\top$	$\top$
1	10	$\top$
2	10	1
3	10	1
4	10	10
5	9	10
6	$\bot$	
7	$\bot$	

Example:



	1		2	
	x	y	x	y
0	$\top$	$\top$	$\top$	$\top$
1	10	$\top$	10	$\top$
2	10	1	$\top$	$\top$
3	10	1	$\top$	$\top$
4	10	10	$\top$	$\top$
5	9	10	$\top$	$\top$
6	$\bot$		$\top$	$\top$
7	$\bot$		$\top$	$\top$

Example:



	1		2		3	
	x	y	x	y	x	y
0	$\top$	$\top$	$\top$	$\top$		
1	10	$\top$	10	$\top$		
2	10	1	$\top$	$\top$		
3	10	1	$\top$	$\top$		
4	10	10	$\top$	$\top$	dito	
5	9	10	$\top$	$\top$		
6	$\bot$		$\top$	$\top$		
7	$\bot$		$\top$	$\top$		

Conclusion:

Although we compute with concrete values, we fail to compute everything :-(

The fixpoint iteration, at least, is guaranteed to terminate:

For n program points and m variables, we maximally need: $n \cdot (m + 1)$ rounds :-)

Caveat:

The effects of edge are not distributive !!!

Counter Example: $f = [\![x = x + y;]\!]^\sharp$

$$
\begin{array}{llcl}
\text{Let} & D_1 & = & \{x \mapsto 2, y \mapsto 3\} \\
 & D_2 & = & \{x \mapsto 3, y \mapsto 2\} \\
\text{Dann} & f\, D_1 \sqcup f\, D_2 & = & \{x \mapsto 5, y \mapsto 3\} \sqcup \{x \mapsto 5, y \mapsto 2\} \\
 & & = & \{x \mapsto 5, y \mapsto \top\} \\
 & & \neq & \{x \mapsto \top, y \mapsto \top\} \\
 & & = & f\, \{x \mapsto \top, y \mapsto \top\} \\
 & & = & f\, (D_1 \sqcup D_2)
\end{array}
$$

:-((

We conclude:

The least solution $\mathcal{D}$ of the constraint system in general yields only an upper approximation of the MOP, i.e.,

$$\mathcal{D}^*[v] \quad \sqsubseteq \quad \mathcal{D}[v]$$

We conclude:

The least solution $\mathcal{D}$ of the constraint system in general yields only an upper approximation of the MOP, i.e.,

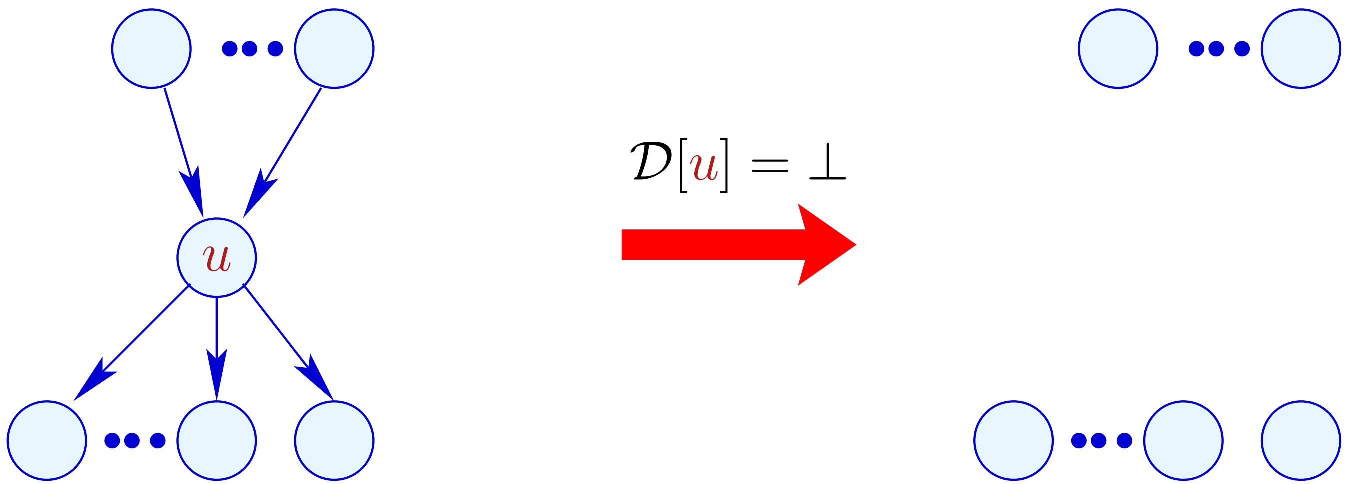
$$\mathcal{D}^*[v] \;\sqsubseteq\; \mathcal{D}[v]$$

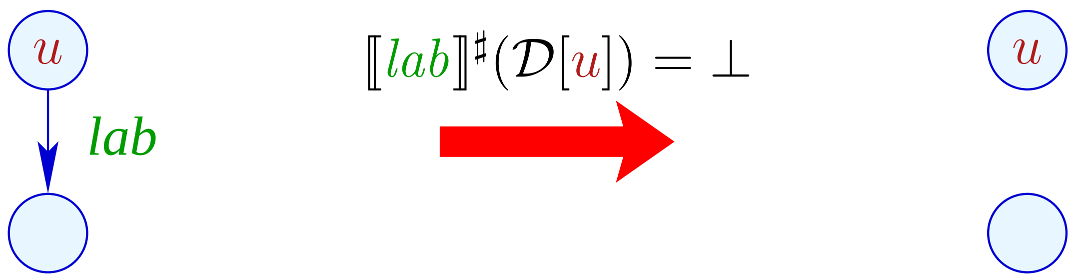
As an upper approximation, $\mathcal{D}[v]$ nonetheless describes the result of every program execution π which reaches v :

$$(\llbracket\pi\rrbracket\,(\rho,\mu)) \;\Delta\; (\mathcal{D}[v])$$

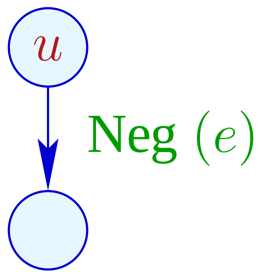
whenever $\llbracket\pi\rrbracket\,(\rho,\mu)$ is defined ;-))

Transformation 4: Removal of Dead Code

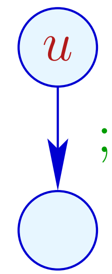


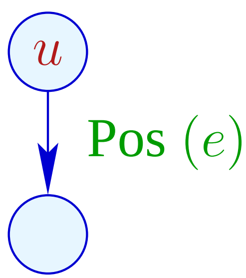


Transformation 4 (cont.): Removal of Dead Code

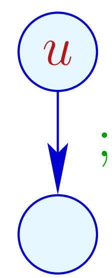


$$\bot \neq \mathcal{D}[u] = D$$
$$[\![e]\!]^\sharp\, D = 0$$

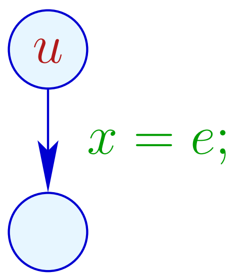




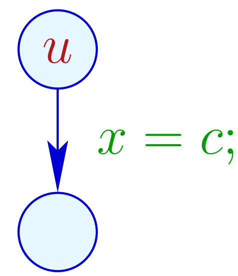
$$\bot \neq \mathcal{D}[u] = D$$
$$[\![e]\!]^\sharp\, D \not\in \{0, \top\}$$



Transformation 4 (cont.): Simplified Expressions



$$\bot \neq \mathcal{D}[u] = D$$
$$[\![e]\!]^\sharp\, D = c$$



Extensions:

- Instead of complete right-hand sides, also subexpressions could be simplified:

$$x + (3 * y) \quad \xRightarrow{\{x \mapsto \top, y \mapsto 5\}} \quad x + 15$$

... and further simplifications be applied, e.g.:

$$
\begin{array}{rcl}
x * 0 & \Longrightarrow & 0 \\
x * 1 & \Longrightarrow & x \\
x + 0 & \Longrightarrow & x \\
x - 0 & \Longrightarrow & x \\
 & \ldots &
\end{array}
$$

- So far, the information of conditions has not yet be optimally exploited:

$$\textsf{if}\ (x == 7)$$
$$y = x + 3;$$

Even if the value of x before the if statement is unknown, we at least know that x definitely has the value 7 — whenever the then-part is entered :-)

Therefore, we can define:

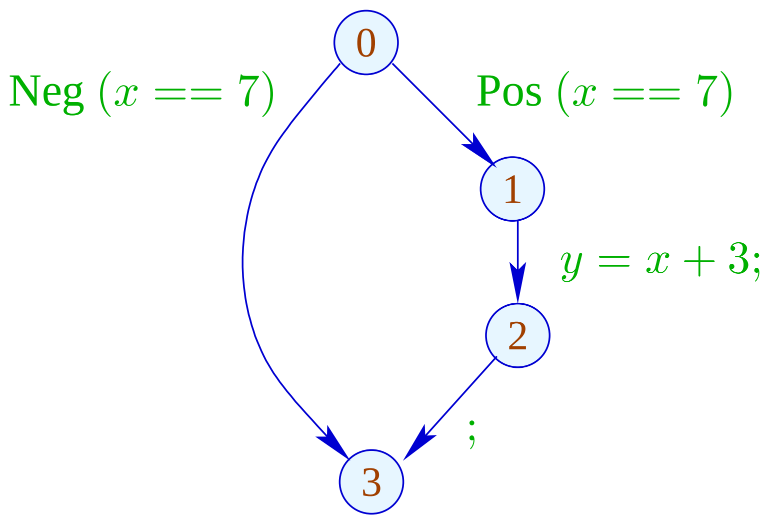
$$[\![\text{Pos}\,(x == e)]\!]^\sharp\, D = \begin{cases} D & \text{if} \quad [\![x == e]\!]^\sharp\, D = 1 \\ \bot & \text{if} \quad [\![x == e]\!]^\sharp\, D = 0 \\ D_1 & \text{otherwise} \end{cases}$$

where

$$D_1 = D \oplus \{x \mapsto (D\,x \sqcap [\![e]\!]^\sharp\, D)\}$$

The effect of an edge labeled $\text{Neg}\,(x \neq e)$ is analogous :-)

Our Example:



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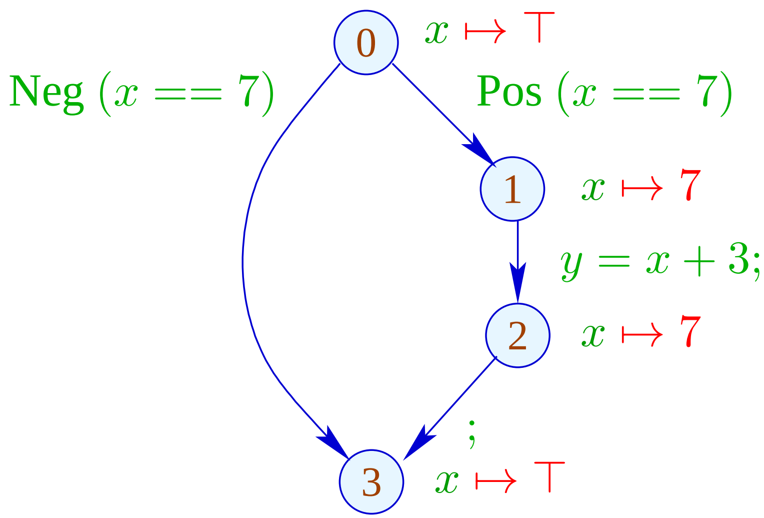

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