Precise Program Analysis, Strategy Iteration and Games

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Part 1: Program Analysis

Part 2: Games

Part 1: Program Analysis

Overview

- Precise Interval Analysis
- Integer Equations
- Strategy Iteration
- Application to Intervals
- Extensions

Interval Analysis

Given:

program variable **x** and program point *u*;

Wanted:

tight interval containing all values of **x** when reaching *u*.

Interval Analysis

Given:

program variable $\mathbf{x}$ and program point u;

Wanted:

tight interval containing all values of $\mathbf{x}$ when reaching u.

$\Longrightarrow$

Program Proving: Certifying absence of overflow;

Program Optimization: Removing array bound checks.

Example

```
for ($i = 0; i < 42; i{+}{+}$)
        if ($0 \leq i \wedge i < 42$){
                $A_1 = A + i$;
                $M[A_1] = i$;
        }
    //   $A$ initial address of an array
    //   array-bound check on $i$
```

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for ($i = 0; i < 42; i{+}{+}$)
        if ($0 \leq i \wedge i < 42$){
                $A_1 = A + i$;
                $M[A_1] = i$;
        }
    //   $A$ initial address of an array
    //   array-bound check on $i$
```

The inner check is superfluous :-)

General Approach [Cousot 1977]

- Represent the collecting semantics of the program by a set of constraints:

$$\begin{array}{lll} \mathcal{C}[\text{start}] & \supseteq & \mathbb{R}^k \\ \mathcal{C}[v] & \supseteq & [\![s]\!](\mathcal{C}[u]) \\ & & (u, s, v) \text{ control-flow edge} \end{array}$$

- Choose a suitable complete lattice $\mathbb{D}$ of abstract values which describe sets of concrete program states ...

General Approach (cont.) [Cousot 1977]

- Represent an abstraction of the collecting semantics of the program also by a set of constraints:

$$\begin{array}{lll} \mathcal{D}[\text{start}] & \sqsupseteq & \top \\ \mathcal{D}[v] & \sqsupseteq & [\![s]\!]^\sharp(\mathcal{D}[u]) \\ & & (u, s, v) \text{ controlflow edge} \end{array}$$

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Challenge:

Compute a sufficiently small solution of this system.

Examples of Abstract Lattices

Affine Equations Karr 1976

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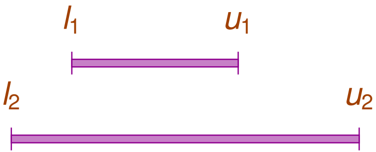
The Lattice of Intervals Cousot 1976

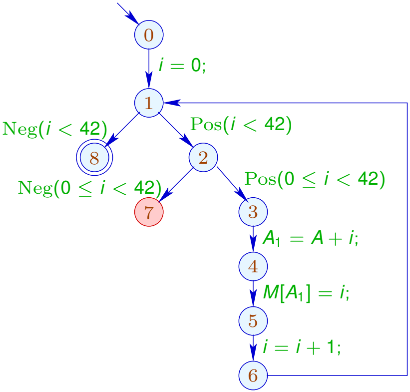
- Ordering:

$$[l_1, u_1] \sqsubseteq [l_2, u_2] \qquad \text{iff} \qquad l_2 \leq l_1 \wedge u_1 \leq u_2$$



- Put up a system of (in-)equations whose least solution approximates the intervals in question !
- Compute some solution !!

... in the Example



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i_2 & = & i_1 \sqcap [-\infty, 41] \\
i_3 & = & i_2 \sqcap [0, 41] \\
i_6 & = & i_3 + [1, 1] \\
i_7 & = & i_2 \sqcap [-\infty, -1] \sqcup \\
 & & \qquad\qquad i_2 \sqcap [42, \infty] \\
i_8 & = & i_1 \sqcap [42, \infty]
\end{array}
$$

Discussion

- The least solution provides only a safe approximation to the concrete intervals :-}
- There are infinite ascending chains, e.g.,

$$\emptyset \sqsubset [0,0] \sqsubset [0,1] \sqsubset [-1,1] \sqsubset [-1,2] \sqsubset \ \ldots$$

Discussion

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$\Longrightarrow$ Naive Kleene iteration may not terminate !

$\Longrightarrow$ Widening for fixpoint acceleration

$\Longrightarrow$ Narrowing as post-processing.

New Developments

polynomially solvable subclass	Su, Wagner, 2004
approximate solution	Costan, Gaubert et al., 2005
revision of pol. case	Reineke, G+S, Wilhelm, 2006 Leroux, 2007
exact least solution	G+S, 2007

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Simplification: Integers

- Lattice:

$$\mathcal{Z} = \{-\infty < \ldots -2 < -1 < 0 < 1 < 2 < \ldots < \infty\}$$

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- Equations $x_j = e_j$ where:

$$e \quad ::= \quad z \mid x_{j'} \mid e_1 + e_2 \mid b \cdot e \mid e_1 \wedge e_2 \mid e_1 \vee e_2$$

for $z \in \mathcal{Z}, b \geq 0$.

Example

$$
\begin{array}{rcl}
i_1 & = & 0 \vee i_6 \\
i_2 & = & i_1 \wedge 41 \\
i_3 & = & i_2 \wedge 41 \\
i_6 & = & i_3 + 1 \\
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i_8 & = & i_1 \wedge \infty
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\\
i_8 & = & i_1 \wedge \infty
\end{array}
$$

$\Longrightarrow$ almost upper half of an interval analysis :-)

Simplification: Disjunctive Systems

$$\begin{aligned} x_1 &= 2 \cdot x_2 \vee -13 \\ x_2 &= x_1 + 5 \vee -7 \\ x_3 &= 3 \cdot x_3 - 6 \vee 4 \end{aligned}$$

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Idea: Bellman-Ford

- Perform n rounds of RR iteration.
- Every variable which then has not yet stabilized receives value ∞ :-)

... in the Example

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	1	2	3	4
x_1	-13	-13	-13	-13
x_2	-7	-7	-7	-7
x_3	4	6	12	∞

The General Case, Idea 1

- Replace any $\wedge$-subexpression $e_1 \wedge e_2$ by one of the e_i $\Longrightarrow$ min strategy
- Solve the resulting disjunctive system
  $\Longrightarrow$ upper bound μ
- Check if μ is a solution of original system.
  Otherwise improve strategy ...

Example

$$\begin{aligned} x_1 &= 2 \cdot x_2 \wedge 1 \\ x_2 &= x_1 \vee -1 \end{aligned}$$

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Approximation:

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Approximation:

$$\begin{array}{rcl} x_1 & = & 2 \cdot x_2 \wedge 1 \\ x_2 & = & x_1 \vee -1 \end{array}$$

has solution: $\mu = \{x_1 \mapsto 1, x_2 \mapsto 1\}$

which is a solution of the original system
— but not the least one :-(

The General Case, Idea 2

- Approximate general systems by conjunctive systems

  ...

- Approximate the least solution by feasible solutions of conjunctive systems :-)

- Compute these by greatest fixpoint iteration :-))

... in the Example

Initial Assignment:

$$\mu_0 = \{x_1 \mapsto -\infty, x_2 \mapsto -\infty\}$$

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Decorated System of Equations:

$$\begin{array}{lll} x_1 & = & 2 \cdot x_2 \wedge 1 \vee -\infty \\ x_2 & = & x_1 \vee -1 \end{array}$$

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Initial Assignment:

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Corresponding Conjunctive System:

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... with solution:

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Corresponding Conjunctive System:

$$\begin{array}{lll} x_1 & = & 2 \cdot x_2 \wedge 1 \vee -\infty \\ x_2 & = & x_1 \vee -1 \end{array}$$

... with greatest solution:

$$\mu_1 = \{x_1 \mapsto -2, x_2 \mapsto -1\}$$

☐

Why it works ...

- We only make profitable switches of decisions.
- The resulting variable assignments are feasible pre-fixpoints.
- Every occurring conjunctive system has exactly one solution exceeding the given pre-fixpoint — which thus is the greatest.

Complexity

$$\mathcal{O}(n \cdot |\mathcal{E}| \cdot (\Pi(m) + n))$$

where:

n :	number of variables
$\vert\mathcal{E}\vert$:	size of the equation system
m :	number of $\vee$
$\Pi(\,.\,)$:	number of strategy iterations

Remark

- The run-time is independent of the sizes of occurring coefficients :-)
- Strategy iteration is a well-known technique in control-theory as well as algorithmic game theory :-)
- Trivially, $\Pi(m) \leq 2^m$:-}
- No example with more than linear iterations has been found so far !!

Application to Interval Equations

- Sub-expressions which evaluate to $\emptyset$ can be eliminated.

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x^+ $==$ upper bound

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- An interval variable x can then be simulated by:

$$\begin{array}{lcr} x^{+} & == & \text{upper bound} \\ x^{-} & == & \text{negated lower bound} \end{array}$$

- Any algorithm for integer systems can also solve interval systems :-))

... in Our Example

$$\begin{array}{lcl}
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First Assumption: all variables have value $\emptyset$ ...

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System for interval bounds:

$$\boxed{\begin{array}{rcl}
i_1^- & = & 0 \\
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results in

$$\mu_1 = \{ i_1 \mapsto [0,0], i_2 \mapsto \emptyset, i_3 \mapsto \emptyset, i_6 \mapsto \emptyset, i_7 \mapsto \emptyset, i_8 \mapsto \emptyset \}$$

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System for interval bounds:

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\begin{array}{lcl}
i_1^- & = & 0 \\
i_2^- & = & i_1^- \wedge \infty \\
i_1^+ & = & 0 \\
i_2^+ & = & i_1^+ \wedge 41
\end{array}
}
$$

since

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System for interval bounds:

$$\boxed{\begin{array}{lcl} i_1^- & = & 0 \\ i_2^- & = & i_1^- \wedge \infty \\ i_1^+ & = & 0 \\ i_2^+ & = & i_1^+ \wedge 41 \end{array}}$$

results in

$$\mu_2 = \{ i_1 \mapsto [0,0], i_2 \mapsto [0,0], i_3 \mapsto \emptyset, i_6 \mapsto \emptyset, i_7 \mapsto \emptyset, i_8 \mapsto \emptyset \}$$

... in Our Example

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System for interval bounds:

$$\boxed{\begin{array}{lcl} i_1^- & = & 0 \wedge i_6^- \\ i_2^- & = & i_1^- \vee 41 \\ i_3^- & = & i_2^- \vee 41 \\ i_6^- & = & i_3^- - 1 \\ \ldots \end{array}}$$

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System for interval bounds:

$$\boxed{\begin{array}{lcl} \ldots & & \\ i_1^+ & = & 0 \vee i_6^+ \\ i_2^+ & = & i_1^+ \wedge 41 \\ i_3^+ & = & i_2^+ \wedge 41 \\ i_6^+ & = & i_3^+ + 1 \end{array}}$$

results in

$$\begin{array}{ll} \mu_3 = \{ & i_1 \mapsto [0, 42], i_2 \mapsto [0, 41], i_3 \mapsto [0, 41], \\ & i_6 \mapsto [0, 42], i_7 \mapsto \emptyset, i_8 \mapsto \emptyset \; \} \end{array}$$

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gives us the final result:

$$\begin{array}{ll} \mu_4 = \{ & i_1 \mapsto [0,42], i_2 \mapsto [0,41], i_3 \mapsto [0,41], \\ & i_6 \mapsto [0,42], i_7 \mapsto \emptyset, i_8 \mapsto [42,42] \; \} \end{array}$$

Extensions

Additionally, we need constructions for:

- negations (easy)
- multiplication (complicated)

Perspectives

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- Techniques can be extended also to deal with equations over rationals and thus solve simple forms of polyhedral analysis!

Perspectives

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- It computes the least solution of the interval constraints precisely, not just an upper bound.
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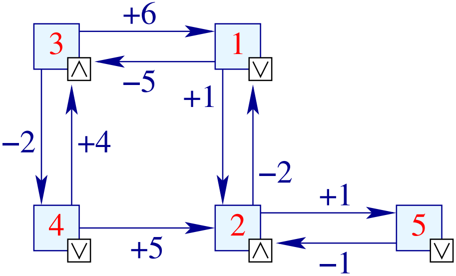
Are interval equations in P?

Part 2: Games

Overview

- Mean Payoff Games
- Total Payoff Games
- Finite Hierarchical Integer Equations
- Simple Integer Equations
- A Lower Bound for Precise Interval Analysis

2-Person-Zero-Sum Infinite Games



Mean Payoff

Play Value for the infinite play $\pi = v_1 v_2 \cdots v_n \cdots$:

$$val(\pi) \;=\; \liminf_{n\to\infty} \frac{\rho\,(v_1, v_2) + \ldots + \rho\,(v_n, v_{n+1})}{n}$$

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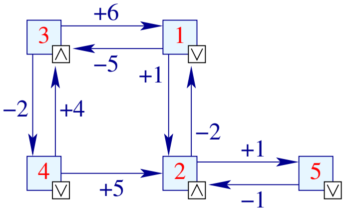

$$val(v_3(v_1 v_2)^\omega) = -\frac{1}{2}$$

Mean Payoff Games

Game Value:

$\mu(v) :=$ maximal play value which
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Property:

Both players have a positional optimal strategy :-)

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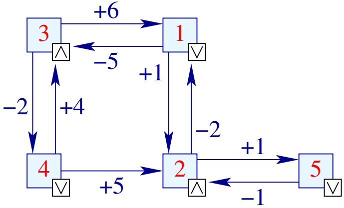

$$val(v_3(v_1 v_2)^\omega) \;=\; -\infty$$
$$val(v_5(v_2 v_1 v_3 v_1)^\omega) \;=\; -8$$

Total Payoff Games

Game Value:

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Property: Zielonka, Gimbert 2004

Both players have a positional optimal strategy :-)

Results

Mean Payoff Games $\Longleftrightarrow$

Total Payoff Games $\Longleftrightarrow$

Finite Hierarchical Systems $\Longleftrightarrow$

Simple Integer Equations

Mean Payoff Through Total Payoff

Lemma

For every position v,

$$\begin{array}{llllll} \mu(v) & < & 0 & \text{iff} & \tau(v) & = -\infty \\ \mu(v) & = & 0 & \text{iff} & \tau(v) & \text{finite} \\ \mu(v) & > & 0 & \text{iff} & \tau(v) & = \infty \end{array}$$

Mean Payoff Through Total Payoff

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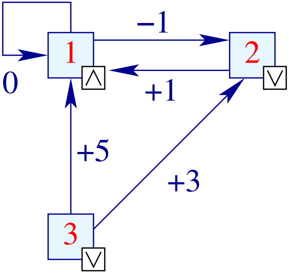
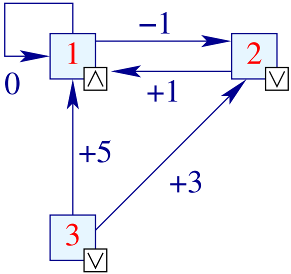
Theorem

Mean-payoff games are polynomially reducible to total-payoff games.

Total Payoff through Equations

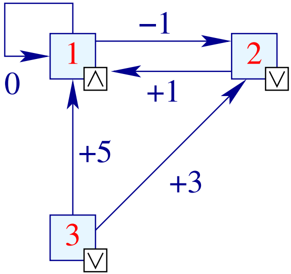


Total Payoff through Equations



$$
\begin{aligned}
x_1 &= x_1 \wedge x_2 - 1 \\
x_2 &= x_1 + 1 \\
x_3 &= x_1 + 5 \vee x_2 + 3
\end{aligned}
$$

Total Payoff through Equations



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\begin{aligned}
x_1 &= y_1 \wedge x_1 \wedge x_2 - 1 \\
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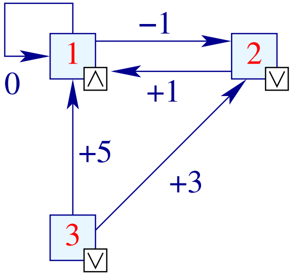
Total Payoff through Equations



$$
\begin{array}{lcl}
y_1 & = & 0 \vee (x_1 \wedge x_2 - 1) \\
y_2 & = & 0 \vee \ \ x_1 + 1 \\
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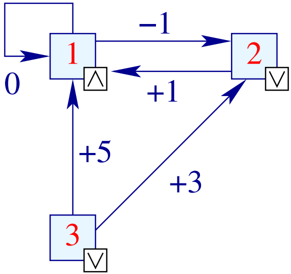

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\begin{aligned}
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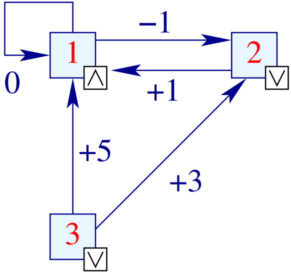

$$
\begin{array}{lcl}
y_1 & = & 0 \vee (x_1 \wedge x_2 - 1) \;\wedge\; 20 \\
y_2 & = & 0 \vee \;\; x_1 + 1 \;\; \wedge\; 20 \\
y_3 & = & 0 \vee (x_1 + 5 \vee x_2 + 3) \wedge 20 \\
x_1 & = & y_1 \;\wedge\;\; x_1 \wedge x_2 - 1 \;\; \vee -3 \\
x_2 & = & y_2 \;\wedge\;\; x_1 + 1 \;\; \vee -3 \\
x_3 & = & y_3 \;\wedge\; (x_1 + 5 \vee x_2 + 3) \;\vee -3
\end{array}
$$

Total Payoff through Equations

- All solutions are finite.
- The outer y_i are least-fixpoint variables whereas the inner x_i are greatest-fixpoint variables!

$\Longrightarrow$ finite hierarchical system

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$\Longrightarrow$ finite hierarchical system

Property: G+S, 2007

Finite hierarchical systems have optimal positional strategies.

Total Payoff through Equations

Theorem

Let ν denote the canonical solution of the hierarchical system. Then

$$\tau(v_i) = \begin{cases} -\infty & \text{if} \quad \nu(x_i) < A \\ \infty & \text{if} \quad \nu(x_i) > B \\ \nu(x_i) & \text{otherwise} \end{cases}$$

for all positions v_i of the total payoff game where:

$$\begin{array}{lcl} A & = & \text{sum of occurring negative values} \\ B & = & \text{sum of occurring positive values} \end{array}$$

Hierarchical through Simple Equations

Idea:

- Nested finite fixpoints over $\mathcal{Z}$ can be read of from a unique fixpoint over an instrumented lattice

G+S, 2007

Hierarchical through Simple Equations

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- The instrumented lattice can be coded into a scaled finite range of integers ...

Hierarchical through Simple Equations

Idea:

- Nested finite fixpoints over $\mathcal{Z}$ can be read of from a unique fixpoint over an instrumented lattice

  G+S, 2007

- The instrumented lattice can be coded into a scaled finite range of integers ...

- Scaling factor: $2 \cdot (n + 1)^d$ where:

  n number of variables
  d number of different ranks of fixpoints

Hierarchical through Simple Equations

... in the Example:

$$y_1 \quad = \quad 0 \vee x_1 \wedge x_2 - 1 \wedge 20$$

scaled with $2 \cdot 7^2 = 98$ becomes:

$$y_1 \quad = \quad 0 \vee x_1 \wedge x_2 - 98 \wedge 1960$$

Hierarchical through Simple Equations

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Hierarchical through Simple Equations

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scaled with $2 \cdot 7^2 = 98$ becomes:

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$$x_1 = (y_1 \wedge x_1 \wedge x_2 - 1) \vee -3$$

scaled with 98 becomes:

$$x_1 = (y_1 \wedge x_1 \wedge x_2 - 98) \vee -294$$

Hierarchical through Simple Equations

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scaled with $2 \cdot 7^2 = 98$ becomes:

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scaled with 98 becomes:

$$x_1 \quad = \quad ((y_1 \wedge x_1 \wedge x_2 - 98) \vee -294) + 7^0$$

Hierarchical through Simple Equations

All together:

$$\begin{array}{lcl} y_1 & = & -7 \vee x_1 - 7 \wedge x_2 - 105 \wedge 1953 \\ y_2 & = & -7 \vee \ \ x_1 + 91 \ \ \wedge 1953 \\ y_3 & = & -7 \vee (x_1 + 483 \vee x_2 + 287) \wedge 1953 \\ \\ x_1 & = & (y_1 + 1 \wedge x_1 + 1 \wedge x_2 - 97) \vee -293 \\ x_2 & = & (y_2 + 1 \wedge x_1 + 99) \vee -293 \\ x_3 & = & (y_3 + 1 \wedge (x_1 + 491 \vee x_2 + 295)) \vee -293 \end{array}$$

Hierarchical through Simple Equations

- All solutions of the resulting system are still finite :-)
- Every simple cycle in the resulting system has cost different from 0

$\Longrightarrow$ simple system of equations

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Theorem

Finite hierarchical systems can be reduced to simple systems of equations.

Simple Equations through Mean Payoff

System:

$$\begin{aligned} x_1 &= 0 \vee x_2 - 7 \\ x_2 &= x_1 + 6 \wedge x_2 - 5 \\ x_3 &= 9 \vee x_2 + 4 \end{aligned}$$

Simple Equations through Mean Payoff

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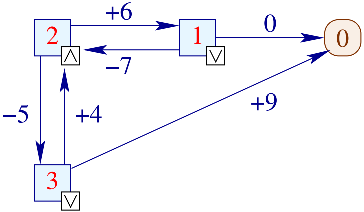
Graphical Representation:



Simple Equations through Mean Payoff

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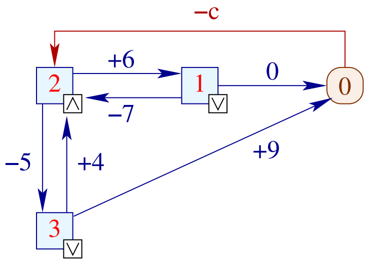
Mean-Payoff Game:



Simple Equations through Mean Payoff

Lemma

Let τ denote the game values of the mean-payoff game with back-edge to x_i with weight $-c$.
Then

$$\nu(x_i) \geq c \quad \text{iff} \quad \tau(x_i) \geq 0$$

Simple Equations through Mean Payoff

Lemma

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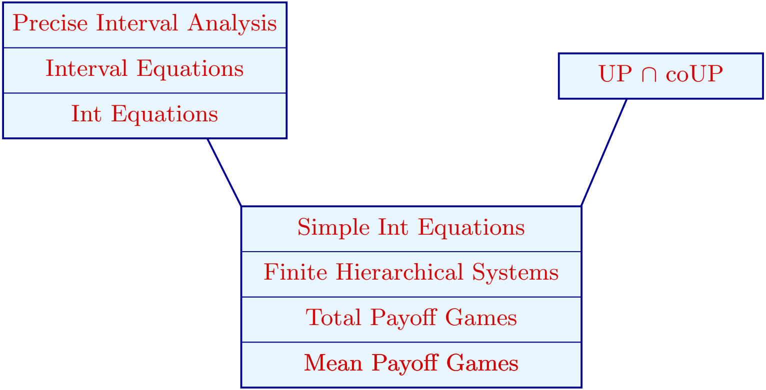
Theorem

Simple equations are polynomially reducible to mean-payoff games.

Summary



Precise Interval Analysis

Interval Equations

Int Equations

UP ∩ coUP

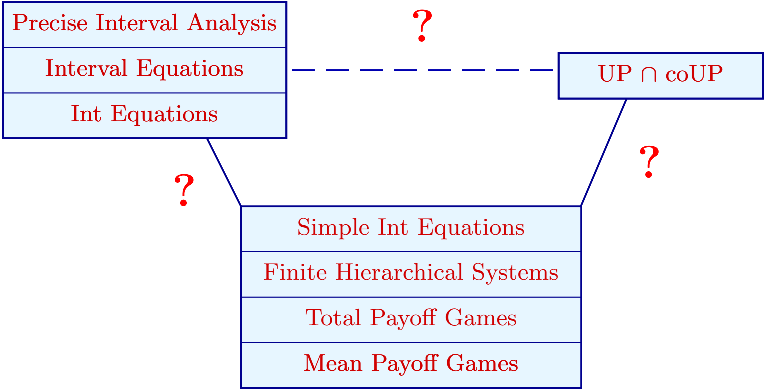
Simple Int Equations

Finite Hierarchical Systems

Total Payoff Games

Mean Payoff Games

Open Problems



Precise Interval Analysis

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