Flat and One-Variable Clauses: Complexity of Verifying Cryptographic Protocols with Single Blind Copying

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Cryptographic protocols with single blind copying were defined and modeled by Comon and Cortier using the new class $\mathcal{C}$ of first order clauses. They showed its satisfiability problem to be in 3-DEXPTIME. We improve this result by showing that satisfiability for this class is NEXPTIME-complete, using new resolution techniques. We show satisfiability to be DEXPTIME-complete if clauses are Horn, which is what is required for modeling cryptographic protocols. While translation to Horn clauses only gives a DEXPTIME upper bound for the secrecy problem for these protocols, we further show that this secrecy problem is actually DEXPTIME-complete.

Categories and Subject Descriptors: D.2.4 [**Software Engineering**]: Software/Program Verification; I.2.3 [**Artificial Intelligence**]: Deduction and Theorem Proving—*Resolution*

General Terms: Security, Verification, Theory

Additional Key Words and Phrases: Cryptographic protocols, first order logic, Horn clauses, resolution, instantiation-based theorem proving

1. INTRODUCTION

Several researchers have pursued modeling of cryptographic protocols using first order clauses [Blanchet 2001; Comon-Lundh and Cortier 2003a; Weidenbach 1999] and related formalisms like tree automata and set constraints[Comon and Cortier 2005; Goubault-Larrecq et al. 2005; Monniaux 1999; Goubault-Larrecq 2000]. While protocol insecurity is NP-complete in case of a bounded number of sessions [Rusinowitch and Turuani 2001], this is helpful only for detecting some attacks. For certifying protocols, the number of sessions cannot be bounded, although we may use other safe abstractions. The approach using first order clauses is particularly useful for this class of problems. A common safe abstraction is to allow a bounded number of nonces, i.e. random numbers, to be used in infinitely many sessions. Security however still remains undecidable [Comon and Cortier 2005]. Hence further restrictions are necessary to obtain decidability.

In this direction, Comon and Cortier [Comon-Lundh and Cortier 2003a; Cortier 2003]

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proposed the notion of protocols with single blind copying. Intuitively this restriction means that agents are allowed to copy at most one piece of data blindly in any protocol step, a restriction satisfied by most protocols in the literature. Comon and Cortier modeled the secrecy problem for these protocols using the new class $\mathcal{C}$ of first order clauses, and showed satisfiability for $\mathcal{C}$ to be decidable [Comon-Lundh and Cortier 2003a] in 3-DEXPTIME [Cortier 2003]. The NEXPTIME lower bound is easy. We show in this paper that satisfiability of this class is in NEXPTIME, thus NEXPTIME-complete. If clauses are restricted to be Horn, which suffices for modeling cryptographic protocols, we show that satisfiability is DEXPTIME-complete (again the lower bound is easy). While translation to clauses only gives a DEXPTIME upper bound for the secrecy problem for this class of protocols, we further show that the secrecy problem for these protocols is also DEXPTIME-complete.

For proving our upper bounds, we introduce several variants of standard ordered resolution with selection and splitting [Bachmair and Ganzinger 2001]. Notably we consider resolution as consisting of instantiation of clauses, and of generation of propositional implications. This is in the style of Ganzinger and Korovin [2003], but we adopt a slightly different approach, and generate interesting implications to obtain optimal complexity. More precisely, while the approach of Ganzinger and Korovin [2003] emphasizes a single phase of instantiation followed by propositional satisfiability checking, we interleave generation of interesting instantiations and propositional implications in an appropriate manner to obtain optimal complexity. We further show how this technique can be employed also in presence of rules for replacement of literals in clauses, which obey some ordering constraints. To deal with the notion of single blind copying we show how terms containing a single variable can be decomposed into simple terms whose unifiers are of very simple forms. As byproducts, we obtain optimal complexity for several subclasses of $\mathcal{C}$, involving so called *flat* and *one-variable* clauses.

Outline: After discussing related work in Section 2, we start in Section 3 by recalling basic notions about first order logic and resolution refinements. In Section 4 we introduce cryptographic protocols with single blind copying, discuss their modeling using the class $\mathcal{C}$ of first order clauses, and show that their secrecy problem is DEXPTIME-hard. To decide the class $\mathcal{C}$ we gradually introduce our techniques by obtaining DEXPTIME-completeness and NEXPTIME-completeness for one-variables clauses and flat clauses in Sections 5 and 6 respectively. In Section 7, the techniques from the two cases are combined with further ideas to show that satisfiability for $\mathcal{C}$ is NEXPTIME-complete. In Section 8 we adapt this proof to show that satisfiability for the Horn fragment of $\mathcal{C}$ is DEXPTIME-complete. We further give an exponential time normalization procedure in the Horn case, which produces a set of simple clauses on which various queries can be efficiently answered.

2. RELATED WORK

For analyzing cryptographic protocols, one line of research is based on the assumption of a bounded number of sessions of the protocol. This is useful for detecting flaws which involve small number of sessions. Rusinowitch and Turuani [2001] showed that in this case, without any further restrictions, protocol-insecurity in NP-complete. A weaker result is presented by Fiore and Abadi [2001] who consider only symmetric cryptography (encryption and decryption using the same key), and present an algorithm that is shown

to be sound but complete only under the assumption of bounded size of keys. Amadio and Lugiez [2000] consider protocols with atomic keys and give a symbolic algorithm to decide whether an erroneous state can be reached.

To guarantee that a protocol has no flaws, one needs to analyze it without bounding the number of sessions. In this case however, Durgin et al. [1999] showed that secrecy is undecidable even with a bound on message size. Further, if the number of nonces is also bounded, then they showed secrecy to be DEXPTIME-complete. The undecidability result of Durgin et al. [1999] is further refined by Amadio and Charatonik [2002] who also consider cryptographic protocols modeled using tail recursive processes. Their approach is to bound the number of parallel sessions, so that the number of nonces in use at any point of time is bounded. These are analyzed using a class of set constraints with a renaming operator.

For analyzing unbounded number of sessions, a common approach is to bound the number of sessions. Also, protocols with unbounded number of sessions can be approximated by protocols with bounded number of sessions, by identifying nonces of different sessions. Under simple assumptions, these approximations are safe in that insecure protocols are approximated by insecure protocols. However even with only only finitely many nonces, with unbounded number of sessions and unbounded message size, secrecy is undecidable [Comon and Cortier 2005]. Hence further restrictions are necessary in order to obtain decidability results. Tree automata and Horn clauses are commonly used for modeling these classes of protocols. The work of Monniaux [1999] was one of the first in this direction. Other work pursuing this approach are mentioned in the introduction.

The class $\mathcal{C}$ of clauses [Comon-Lundh and Cortier 2003a; Cortier 2003] is also closely related to the class of tree automata with one memory Comon et al. [2001; Comon and Cortier [2005]. This work also deals with a related class of set constraints for modeling cryptographic protocols.

Use of automated deduction techniques for deciding fragments of first-order logic has been extensively studied. Maslov [1964] defined the inverse method, which is now well-understood as a form of resolution, and claimed that it provides decision procedures for several classes. Joyner Jr. [1976] used ordered resolution for deciding several classes. Ferm¨uller et al. [2001] also provide several other examples of classes which can be decided by resolution techniques.

While the class we study does not include the equality relation, superposition or paramodulation calculi have been studied to deal with the equality relation. For example, Bachmair et al. [1993b] show that the monadic class with equality can be decided using these techniques.

There are also many similarities between some classes of clauses, in particular flat clauses, and some classes of set constraints [Bachmair et al. 1993a; Goubault-Larrecq 2002]. It is traditional to decide both clause sets and set constraints by some saturation procedures, although there are notable differences. The main difference lies in the last normalization rule introduced in Section 8.1 for states corresponding to intersections of several other states. This is a typical rule encountered in saturating set constraints, see [Charatonik and Podelski 1997] for example, but is rarely seen in saturating clause sets using resolution.

Our idea of resolution modulo propositional reasoning, introduced in Section 6, involves instantiations of clauses and generation of propositional implications in a suitable way to hopefully generate fewer clauses than usual resolution techniques would generate. It

has long been recognized that, while resolution handles first-order phenomena efficiently through unification of terms, it is very inefficient at propositional reasoning. The first to have observed this, and to have proposed a cure, are Lee and Plaisted [1992] who proposed the idea of hyper-linking of clauses. Their prover CLIN is based on these ideas. Goubault's use of BDDs at first order can be seen as a variant of this idea [Goubault 1994]. Ganzinger and Korovin [2003], cited in the introduction, also pursue the idea of generating suitable instantiations of clauses and then checking propositional unsatisfiability. To our knowledge, none of these work address decidability issues.

3. RESOLUTION

We recall standard notions from first order logic. Fix a signature Σ of function symbols each with a given arity, and containing at least one zero-ary symbol. Let r be the maximal arity of function symbols in Σ. Fix a set $\mathbf{X} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots\}$ of variables. Note that $\mathbf{x}_1, \mathbf{x}_2, \ldots$ (in bold face) are the actual elements of $\mathbf{X}$, where as $x, y, z, x_1, y_1, \ldots$ are used to represent arbitrary elements of $\mathbf{X}$. The set $T_\Sigma(\mathbf{X})$ of terms built from Σ and $\mathbf{X}$ is defined as usual. T_Σ is the set of *ground terms*, i.e. those not containing any variables. *Atoms* A are of the form $P(t_1, \ldots, t_n)$ where P is an n-ary predicate and t_i's are terms. *Literals* L are either positive literals $+A$ (or simply A) or negative literals $-A$, where A is an atom. $-(-A)$ is another notation for A. $\pm$ denotes $+$ or $-$ and $\mp$ denotes the opposite sign (and similarly for notations $\pm', \mp', \ldots$). A *clause* is a finite set of literals. A *negative clause* is one which contains only negative literals. If M is any term, literal or clause then the set $\mathsf{fv}(M)$ of variables occurring in them is defined as usual. If C_1 and C_2 are clauses then $C_1 \vee C_2$ denotes $C_1 \cup C_2$. $C \vee \{L\}$ is written as $C \vee L$ (In this notation, we allow the possibility of $L \in C$). If $C_1, \ldots, C_n$ are clauses such that $\mathsf{fv}(C_i) \cap \mathsf{fv}(C_j) = \emptyset$ for $i \neq j$, and if C_i is non-empty for $i \geq 2$, then the clause $C_1 \vee \ldots \vee C_n$ is also written as $C_1 \sqcup \ldots \sqcup C_n$ to emphasize this property. *Ground literals and clauses* are ones not containing variables. A term, literal or clause is *trivial* if it contains no function symbols. A substitution is a function $\sigma : \mathbf{X} \to T_\Sigma(\mathbf{X})$. *Ground substitutions* map every variable to a ground term. We write $\sigma = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$ to say that $x_i\sigma = t_i$ for $1 \leq i \leq n$ and $x\sigma$ is some arbitrary term (whose value will not be relevant in the context) for $x \notin \{x_1, \ldots, x_n\}$. If M is a term, literal, clause, substitution or set of such objects, then the effect $M\sigma$ of applying σ to M is defined as usual. *Renamings* are bijections $\sigma : \mathbf{X} \to \mathbf{X}$. If M is a term, literal, clause or substitution, then a renaming of M is of the form $M\sigma$ for some renaming σ, and an instance of M is of the form $M\sigma$ for some substitution σ. If M and N are terms or literals then a *unifier* of M and N is a substitution σ such that $M\sigma = N\sigma$. If such a unifier exists then there is also a *most general unifier (mgu)*, i.e. a unifier σ such that for every unifier σ' of M and N, there is some σ'' such that $\sigma' = \sigma\sigma''$. Most general unifiers are unique upto renaming: if σ_1 and σ_2 are two mgus of M and N then σ_1 is a renaming of σ_2. Hence we may use the notation $mgu(M, N)$ to denote one of them without ambiguity. We write $M[x_1, \ldots, x_n]$ to say that $\mathsf{fv}(M) \subseteq \{x_1, \ldots, x_n\}$. If $t_1, \ldots, t_n$ are terms then $M[t_1, \ldots, t_n]$ denotes $M\{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$. If N is a set of terms then $M[N] = \{M[t_1, \ldots, t_n] \mid t_1, \ldots, t_n \in N\}$. If M is a set of terms, atoms, literals or clauses then $M[N] = \bigcup_{m \in M} m[N]$. A *Herbrand interpretation* $\mathcal{H}$ is a set of ground atoms. A clause C is *satisfied* in $\mathcal{H}$ if for every ground substitution σ there is some A such that, either $A \in \mathcal{H}$ and $A \in C\sigma$, or $A \notin \mathcal{H}$ and $-A \in C\sigma$. A set S of clauses is satisfied in $\mathcal{H}$ if every clause of S is satisfied in $\mathcal{H}$. If such an $\mathcal{H}$ exists then

S is *satisfiable*, and $\mathcal{H}$ is a *Herbrand model* of S. A *Horn clause* is one containing at most one positive literal. If a set of Horn clauses is satisfiable then it has a least Herbrand model w.r.t. the subset ordering. A *definite clause* is one containing exactly one positive literal. The clause $A \vee -A_1 \vee \ldots \vee -A_n$ is also written using the Prolog-like notation $A \Leftarrow A_1 \wedge \ldots \wedge A_n$. A is called the *head* and the remaining part the *body* of the clause. A set of definite clauses always has a least Herbrand model $\mathcal{H}$ defined inductively using the rule that if $A \Leftarrow A_1 \wedge \ldots \wedge A_n$ is present in the clause set, if σ is a ground substitution, if each $A_i\sigma \in \mathcal{H}$ then $A\sigma \in \mathcal{H}$. Hence the presence of a ground atom A' in $\mathcal{H}$ can be justified by a tree like structure consisting of using a clause and a ground substitution at each node. We call this the *derivation* of A'. A' is said to be *derivable* or *reachable*, which is equivalent to saying that $S \cup \{-A\}$ is unsatisfiable.

Resolution and its refinements are well known methods for testing unsatisfiability of sets of clauses. Given a strict partial order $<$ on atoms, a literal $\pm A$ is *maximal* in a clause C if there is no literal $\pm' B \in C$ with $A < B$. *Binary ordered resolution* and *ordered factorization* w.r.t. ordering $<$ are defined by the following two rules respectively:

$$\frac{C_1 \vee A \quad -B \vee C_2}{C_1\sigma \vee C_2\sigma} \qquad \frac{C_1 \vee \pm A \vee \pm B}{C_1\sigma \vee A\sigma}$$

where $\sigma = mgu(A, B)$ in both rules, A and B are maximal in the left and right premises respectively of the first rule, and A and B are both maximal in the premise of the second rule. We rename the premises of the first rule before resolution so that they do not share variables. The ordering $<$ is *stable* if: whenever $A_1 < A_2$ then $A_1\sigma < A_2\sigma$ for all substitutions σ. We write $S \Rightarrow_< S \cup \{C\}$ to say that C is obtained by one application of the binary ordered resolution or binary factorization rule on clauses in S (the subscript denotes the ordering used).

Another inference rule is *splitting*. This can be described using *tableaux*. A *tableau* is of the form $S_1 \mid \ldots \mid S_n$, where $n \geq 0$ and each S_i, called a *branch* of the tableau, is a set of clauses (the $\mid$ operator is associative and commutative). A tableau is *satisfiable* if at least one of its branches is satisfiable. The tableau is called *closed* if each S_i contains the empty clause, denoted $\Box$. The *splitting* step on tableaux is defined by the rule

$$\mathcal{T} \mid S \rightarrow_{spl} \mathcal{T} \mid (S \setminus \{C\}) \cup \{C_1\} \mid \ldots \mid (S \setminus \{C\}) \cup \{C_n\}$$

whenever $C = C_1 \sqcup \ldots \sqcup C_n \in S$, each C_i is non-empty. C_i are called *components* of the clause C being split. It is well known that splitting preserves satisfiability of tableaux. We may choose to apply splitting eagerly, or lazily or in some other fashion. Hence we define a *splitting strategy* to be a function ϕ such that $\mathcal{T} \rightarrow_{spl} \phi(\mathcal{T})$ for all tableaux $\mathcal{T}$. The relation $\Rightarrow_<$ is extended to tableaux as expected. Ordered resolution with splitting strategy is then defined by the rule

$$\mathcal{T}_1 \Rightarrow_{<,\phi} \phi(\mathcal{T}_2) \text{ whenever } \mathcal{T}_1 \Rightarrow_< \mathcal{T}_2$$

This provides us with a well known sound and complete method for testing satisfiability. For any binary relation R, R^* denotes the reflexive transitive closure of R, and R^+ denotes the transitive closure of R.

LEMMA 3.1. *For any set S of clauses, for any stable ordering $<$, and for any splitting strategy ϕ, S is unsatisfiable iff $S \Rightarrow^*_{<,\phi} \mathcal{T}$ for some closed $\mathcal{T}$.*

If all predicates are zero-ary then the resulting clauses are *propositional clauses*. In this case we write $S \models_{\mathrm{p}} T$ to say that every Herbrand model of S is a Herbrand model of T. This notation will also be used when S and T are sets of first order clauses, by treating every (ground or non-ground) atom as a zero-ary predicate. For example $\{P(a), -P(a)\} \models_{\mathrm{p}} \Box$ but $\{P(x), -P(a)\} \not\models_{\mathrm{p}} \Box$. $S \models_{\mathrm{p}} \{C\}$ is also written as $S \models_{\mathrm{p}} C$. If $S \models_{\mathrm{p}} C$ then clearly $S\sigma \models_{\mathrm{p}} C\sigma$ for every substitution σ.

4. CRYPTOGRAPHIC PROTOCOLS

We assume that Σ contains the binary functions $\{\_\}\_$ and $\langle \_, \_ \rangle$ denoting encryption and pairing. *Messages* are terms of $T_\Sigma(\mathbf{X})$. A *state* is of the form $S(M_1, \ldots, M_n)$ where S with arity n is from a finite set of *control points* and M_i are messages. It denotes an agent at control point S with messages M_i in its memory. An *initialization state* is a state not containing variables. We assume some strict partial order $<$ on the set of control points. A *protocol rule* is of the form

$$S_1(M_1, \ldots, M_m) : \mathsf{recv}(M) \rightarrow S_2(N_1, \ldots, N_n) : \mathsf{send}(N)$$

where $S_1 < S_2$, M_i, N_j are messages, and M and N are each either a message, or a dummy symbol ? indicating nothing is received (resp. sent). For secrecy analysis we can replace ? by some public message, i.e. one which is known to everyone including the adversary. The rule says that an agent in state $S_1(M_1, \ldots, M_m)$ can receive message M, send a message N, and then move to state $S_2(N_1, \ldots, N_n)$, thus also modifying the messages in its memory. A *protocol* is a finite set of initialization states and protocol rules. This model is in the style of [Durgin et al. 1999] and [Comon and Cortier 2005]. The assumption of single blind copying then says that each protocol rule contains at most one variable (which may occur anywhere any number of times in that rule). For example, the public-key Needham-Schroeder protocol

$$\begin{array}{l} A \rightarrow B : \ \{A, N_A\}_{K_B} \\ B \rightarrow A : \ \{N_A, N_B\}_{K_B} \\ A \rightarrow B : \ \{N_B\}_{K_B} \end{array}$$

is written in our notation as follows. Firstly, a bounded number of agents suffice for finding all attacks against secrecy [Comon-Lundh and Cortier 2003b; 2003a], under very reasonable assumptions which are satisfied by this protocol. Note that the argument of [Comon-Lundh and Cortier 2003b] is in a somewhat different model, but is easily adapted to different models. Now, for every pair of agents A and B in our system we have two nonces N^1_{AB} and N^2_{AB} to be used in sessions where A plays the initiator's role and B plays the responder's role. We have initialization states $\mathsf{Init}_0(A, N^1_{AB})$ and $\mathsf{Resp}_0(B, N^2_{AB})$ for all agents A and B. Note that A, N^1_{AB} etc. are constants, contrary to the Prolog convention which uses identifiers starting with capitals for variables. Corresponding to the three lines in the protocol we have rules for all agents A and B

$$\begin{array}{rcl} \mathsf{Init}_0(A, N^1_{AB}){:}\mathsf{recv}(?) & \rightarrow & \mathsf{Init}_1(A, N^1_{AB}){:}\mathsf{send}(\{\langle A, N^1_{AB}\rangle\}_{K_B}) \\ \mathsf{Resp}_0(B, N^2_{AB}){:}\mathsf{recv}(\{\langle A, x\rangle\}_{K_B}) & \rightarrow & \mathsf{Resp}_1(B, x, N^2_{AB}){:}\mathsf{send}(\{\langle x, N^2_{AB}\rangle\}_{K_A}) \\ \mathsf{Init}_1(A, N^1_{AB}){:}\mathsf{recv}(\{\langle N^1_{AB}, x\rangle\}_{K_A}) & \rightarrow & \mathsf{Init}_2(A, N^1_{AB}, x){:}\mathsf{send}(\{x\}_{K_B}) \\ \mathsf{Resp}_1(B, x, N^2_{AB}){:}\mathsf{recv}(\{N^2_{AB}\}_{K_B}) & \rightarrow & \mathsf{Resp}_2(B, x, N^2_{AB}){:}\mathsf{send}(?) \end{array}$$

Any initialization state can be created any number of times and any protocol rule can be executed any number of times. The adversary has full control over the network: all

messages received by agents are actually sent by the adversary and all messages sent by agents are actually received by the adversary. The adversary can obtain new messages from messages he knows, e.g. by performing encryption and decryption. To model this using Horn clauses, we create a unary predicate reach to model reachable states, and a unary predicate known to model messages known to the adversary. The initialization state $S(M_1, \ldots, M_n)$ is then modeled by the clause $\mathsf{reach}(S(M_1, \ldots, M_n))$, where S is a new function symbol we create. The protocol rule

$$S_1(M_1, \ldots, M_m) : \mathsf{recv}(M) \rightarrow S_2(N_1, \ldots, N_n) : \mathsf{send}(N)$$

is modeled by the clauses

$$\mathsf{known}(N) \vee -\mathsf{reach}(S_1(M_1, \ldots, M_m)) \vee -\mathsf{known}(M)$$
$$\mathsf{reach}(S_2(N_1, \ldots, N_n)) \vee -\mathsf{reach}(S_1(M_1, \ldots, M_m)) \vee -\mathsf{known}(M)$$

Under the assumption of single blind copying it is clear that all these clauses are *one-variable clauses*, i.e. clauses containing at most one variable. We need further clauses to express adversary capabilities. The clauses

$$\mathsf{known}(\{\mathbf{x}_1\}_{\mathbf{x}_2}) \vee -\mathsf{known}(\mathbf{x}_1) \vee -\mathsf{known}(\mathbf{x}_2)$$
$$\mathsf{known}(\mathbf{x}_1) \vee -\mathsf{known}(\{\mathbf{x}_1\}_{\mathbf{x}_2}) \vee -\mathsf{known}(\mathbf{x}_2)$$

express the encryption and decryption abilities of the adversary. We have similar clauses for his pairing and unpairing abilities, as well as clauses

$$\mathsf{known}(f(\mathbf{x}_1, \ldots, \mathbf{x}_n)) \vee -\mathsf{known}(\mathbf{x}_1) \vee \ldots \vee -\mathsf{known}(\mathbf{x}_n)$$

for any function f that the adversary knows to apply. All these are clearly *flat clauses*, i.e. clauses of the form

$$C = \bigvee_{i=1}^{k} \pm_i P_i(f_i(x_1^i, \ldots, x_{n_i}^i)) \vee \bigvee_{j=1}^{l} \pm_j Q_j(x_j)$$

where $\{x_1^i, \ldots, x_{n_i}^i\} = \mathsf{fv}(C)$ for $1 \leq i \leq k$. Asymmetric keys, i.e. keys K such that message $\{M\}_K$ can only be decrypted with the inverse key K^{-1}, are also easily dealt with using flat and one-variable clauses. The adversary's knowledge of other data c like agent's names, public keys, etc are expressed by clauses $\mathsf{known}(c)$. Then the least Herbrand model of this set of clauses describes exactly the reachable states and the messages known to the adversary. Then to check whether some message M remains secret, we add the clause $-\mathsf{known}(M)$ and check whether the resulting set is satisfiable.

A set of clauses is in the class $\mathcal{V}_1$ if each of its members is a one-variable clause. A set of clauses is in the class $\mathcal{F}$ if each of its members is a flat clause. More generally we have the class $\mathcal{C}$ proposed by Comon and Cortier [Comon-Lundh and Cortier 2003a; Cortier 2003]: a set of clauses S is in the class $\mathcal{C}$ if for each $C \in S$ one of the following conditions is satisfied.

(1) C is a one-variable clause
(2) $C = \bigvee_{i=1}^{k} \pm_i P_i(u_i[f_i(x_1^i, \ldots, x_{n_i}^i)]) \vee \bigvee_{j=1}^{l} \pm_j Q_j(x_j)$, where for $1 \leq i \leq k$ we have $\{x_1^i, \ldots, x_{n_i}^i\} = \mathsf{fv}(C)$ and u_i contains at most one variable.

If all clauses are Horn then we have the corresponding classes $\mathcal{V}_1 Horn$, $\mathcal{F} Horn$ and $\mathcal{C} Horn$. Clearly the classes $\mathcal{V}_1$ (resp. $\mathcal{V}_1 Horn$) and $\mathcal{F}$ (resp. $\mathcal{F} Horn$) are included in the

class $\mathcal{C}$ (resp. $\mathcal{C}Horn$) since the u_i's above can be trivial. Conversely any clause set in $\mathcal{C}$ can be considered as containing just flat and one-variable clauses. This is because we can replace a clause $C \vee \pm P(u[f(x_1, \ldots, x_n)])$ by the clause $C \vee \pm Pu(f(x_1, \ldots, x_n))$ and add clauses $-Pu(x) \vee P(u[x])$ and $Pu(x) \vee -P(u[x])$ where Pu is a fresh predicate. This transformation takes polynomial time and preserves satisfiability of the clause set. Hence now we need to deal with just flat and one-variable clauses. In the rest of the paper we derive optimal complexity results for all these classes.

Still this only gives us an upper bound for the secrecy problem of protocols since the clauses could be more general than necessary. It turns out, however, that this is not the case. In order to show this we rely on a reduction of the reachability problem for *alternating pushdown systems (APDS)*. In form of Horn clauses, an *APDS* is a finite set of clauses of the form

(i) $P(a)$ where a is a zero-ary symbol

(ii) $P(s[x]) \vee -Q(t[x])$ where s and t involve only unary function symbols, and

(iii) $P(x) \vee -P_1(x) \vee -P_2(x)$

Reachability in APDS is DEXPTIME-hard [Chandra et al. 1981]. We encode this problem into secrecy of protocols, as in [Durgin et al. 1999]. Let K be a (symmetric) key not known to the adversary. Encode atoms $P(t)$ as messages $\{\langle P, t\rangle\}_K$, by treating P as some data. Create initialization states S_1 and S_2 (no message is stored in the states). Clauses (i-iii) above are translated as

$$\begin{array}{ll} S_1 : \mathsf{recv}(?) & \rightarrow S_2 : \mathsf{send}(\{\langle P, a\rangle\}_K) \\ S_1 : \mathsf{recv}(\{\langle Q, t[x]\rangle\}_K) & \rightarrow S_2 : \mathsf{send}(\{\langle P, s[x]\rangle\}_K) \\ S_1 : \mathsf{recv}(\langle\{\langle P_1, x\rangle\}_K, \{\langle P_2, x\rangle\}_K\rangle) & \rightarrow S_2 : \mathsf{send}(\{\langle P, x\rangle\}_K) \end{array}$$

The intuition is that the adversary cannot decrypt messages encrypted with K. He also cannot encrypt messages with K. He can only forward messages which are encrypted with K. However he has the ability to pair messages. This is utilized in the translation of clause (iii). Then a message $\{M\}_K$ is known to the adversary iff M is of the form $\langle P, t\rangle$ and $P(t)$ is reachable in the APDS.

THEOREM 4.1. *Secrecy problem for cryptographic protocols with single blind copying, with bounded number of nonces but unbounded number of sessions is DEXPTIME-hard, even if no message is allowed to be stored at any control point.*

We make a few remarks regarding this model of cryptographic protocols. The above encoding of APDS requires very simple clauses, hence the generality of the shape of clauses and protocol rules comes at no increased price in terms of complexity. The model is also general enough. The single blind copying assumption is a natural one observed in many existing protocols. While the assumption of finitely many nonces is not satisfied by most protocols, it is still reasonable to make this assumption. Firstly, as mentioned above, we need to consider only bounded number of agents for analyzing protocols. Hence although nonces need to be modeled as functions of pairs of agents, this does not lead to infinitely many nonces. However there could still be infinitely may nonces due to the number of sessions. Then one can use safe approximations which to obtain only finitely many nonces. These approximations are safe in that insecure protocols remain insecure.

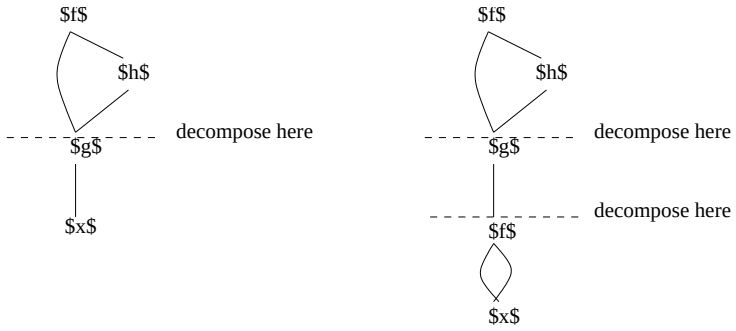


Fig. 1. Decomposition of one-variable terms

5. ONE VARIABLE CLAUSES: DECOMPOSITION OF TERMS

We first show that satisfiability for the classes $\mathcal{V}_1$ and $\mathcal{V}_1 Horn$ is DEXPTIME-complete. We recall also that although we consider only unary predicates, this is no restriction in the case of one-variable clauses, since we can encode atoms $P(t_1, \ldots, t_n)$ as $P'(f_n(t_1 \ldots, t_n))$ for fresh P' and f_n for every P of arity n. As shown in [Comon-Lundh and Cortier 2003a; Cortier 2003], ordered resolution on one-variable clauses, for a suitable ordering, leads to a linear bound on the height of terms produced. This does not suffice for obtaining a DEXPTIME upper bound and we need to examine the forms of unifiers produced during resolution. We consider terms containing at most one variable (call them *one-variable terms*) to be compositions of simpler terms. A non-ground one-variable term $t[x]$ is called *irreducible* if it is not of the form $u[v[x]]$ for any non-ground non-trivial one-variable terms $u[x]$ and $v[x]$. The term $f(g(x), h(g(x)))$ for example is not irreducible because it can be written as $f(x, h(x))[g(x)]$. The term $f(g(f(x, x)), h(g(f(x, x))))$ is not irreducible because it can be written as f(x,h(x))[g(x) [f(x,x)]]. Figure 1 provides some intuition. The term $f'(x, g(x), a)$ is irreducible. Unifying it with the irreducible term $f'(h(y), g(h(a)), y)$ produces ground unifier $\{x \mapsto h(y)[a], y \mapsto a\}$ and both $h(y)$ and a are strict subterms of the given terms. Indeed we find:

LEMMA 5.1. *Let $s[x]$ and $t[y]$ be irreducible, non-ground and non-trivial terms where $x \neq y$ and $s[x] \neq t[x]$. If s and t have a unifier σ then $x\sigma, y\sigma \in U[V]$ where U is the set of non-ground (possibly trivial) strict subterms of s and t, and V is the set of ground strict subterms of s and t.*

PROOF. See Appendix A. □

In case both terms (even if not irreducible) have the same variable we have the following easy result:

LEMMA 5.2. *Let σ be a unifier of two non-trivial, non-ground and distinct one-variable terms $s[x]$ and $t[x]$. Then $x\sigma$ is a ground strict subterm of s or of t.*

PROOF. See Appendix A. □

In the following one-variable clauses are simplified to involve only irreducible terms.

LEMMA 5.3. *Any non-ground one-variable term $t[x]$ can be uniquely written as $t[x] = t_1[t_2[\ldots[t_n[x]]\ldots]]$ where $n \geq 0$ and each $t_i[x]$ is non-trivial, non-ground and irreducible. This decomposition can be computed in time polynomial in the size of t.*

PROOF. We represent $t[x]$ as a DAG by doing maximal sharing of subterms. If $t[x] = x$ then the result is trivial. Otherwise let N be the position in this graph, other than the root node, closest to the root such that N lies on every path from the root to the node corresponding to the subterm x. Let t' be the strict subterm of t at position N and let t_1 be the term obtained from t by replacing the sub-DAG at N by x. Then $t = t_1[t']$ and t_1 is irreducible. We then recursively decompose t'.

Uniqueness of decomposition follows from Lemma 5.1 according to the following argument. If decomposition were not unique then let $t[x] = t'_1[t'_2[\ldots[t'_m[x]]\ldots]]$ be another decomposition. By symmetry one of the following cases occur.

—$n < m$ and $t_i[x] = t'_i[x]$ for $1 \leq i \leq n$.
But then $t_1[t_2[\ldots[t_n[x]]\ldots]] \neq t'_1[t'_2[\ldots[t'_m[x]]\ldots]]$, leading to contradiction.

—We have some k such that $t_i[x] = t'_i[x]$ for $1 \leq i < k$, and $t_k[x] \neq t'_k[x]$. We must have $t_k[\ldots[t_n[x]]\ldots] = t'_k[\ldots[t'_m[x]]\ldots]$. Hence the terms $t_k[x]$ and $t'_k[y]$ have a unifier $\{x \mapsto t_{k+1}[\ldots[t_m[x]]\ldots], y \mapsto t'_{k+1}[\ldots[t_m[x]]\ldots]$, which leads to a contradiction by Lemma 5.1.

□

The above result is a cornerstone of the rest of this paper. It illustrates an important property of one-variable terms. These terms can be thought of as strings of symbols, by considering an irreducible term as analogous to a symbol. The analogy is not exact though, since distinct one-variables terms can still unify, as explained by Lemmas 5.1 and 5.2. But the unification fortunately always produces ground terms.

Above and elsewhere, if $n = 0$ then $t_1[t_2[\ldots[t_n[x]]\ldots]]$ denotes x. Now if a clause set contains a clause $C = C' \vee \pm P(t[x])$, with $t[x]$ being non-ground, if $t[x] = t_1[\ldots[t_n[x]]\ldots]$ where each t_i is non-trivial and irreducible, then we create fresh predicates $Pt_1 \ldots t_i$ for $1 \leq i \leq n-1$ and replace C by the clause $C' \vee \pm Pt_1 \ldots t_{n-1}(t_n[x])$. Also we add clauses $Pt_1 \ldots t_i(t_{i+1}[x]) \vee -Pt_1 \ldots t_{i+1}(x)$ and $-Pt_1 \ldots t_i(t_{i+1}[x]) \vee Pt_1 \ldots t_{i+1}(x)$ for $0 \leq i \leq n-2$ to our clause set. Note that the predicates $Pt_1 \ldots t_i$ are considered invariant under renaming of terms t_j. For $i = 0$, $Pt_1 \ldots t_i$ is the same as P. Our transformation preserves satisfiability of the clause set. By Lemma 5.3 this takes polynomial time and eventually all non-ground literals in clauses are of the form $\pm P(t)$ with irreducible t. Next if the clause set is of the form $S \cup \{C_1 \cup C_2\}$, where C_1 is non-empty and has only ground literals, and C_2 is non-empty and has only non-ground literals, then we do splitting to produce $S \cup \{C_1\} \mid S \cup \{C_2\}$. This process produces at most exponentially many branches each of which has polynomial size. Now it suffices to decide satisfiability of each branch in DEXPTIME. Hence now we assume that each clause is either:

(**Ca**) a ground clause, or

(**Cb**) a clause containing exactly one variable, each of whose literals is of the form $\pm P(t[x])$ where t is non-ground and irreducible.

Consider a set S of clauses of type Ca and Cb. We show how to decide satisfiability of the set S. Wlog we assume that all clauses in S of type Cb contain the variable $\mathbf{x}_1$. Let Ng be the set of non-ground terms $t[\mathbf{x}_1]$ occurring as arguments in literals in S. Let Ngs be the set of non-ground subterms $t[\mathbf{x}_1]$ of terms in Ng. We assume that Ng and Ngs always contain the trivial term $\mathbf{x}_1$, otherwise we add this term to both sets. Let G be the set of ground subterms of terms occurring as arguments in literals in S. The sizes of $\mathsf{Ng}, \mathsf{Ngs}$ and G are polynomial. Let $S^\dagger$ be the set of clauses of type Ca and Cb which only contain literals of

the form $\pm P(t)$ for some $t \in \mathsf{Ng} \cup \mathsf{Ng}[\mathsf{Ngs}[\mathsf{G}]]$ (observe that $\mathsf{G} \subseteq \mathsf{Ngs}[\mathsf{G}] \subseteq \mathsf{Ng}[\mathsf{Ngs}[\mathsf{G}]]$). Ng, Ngs and G are of polynomial size, hence $\mathsf{Ng}[\mathsf{Ngs}[\mathsf{G}]]$ is of polynomial size. The size of $S^\dagger$ is at most exponential.

For resolution we use ordering $\prec$: $P(s) \prec Q(t)$ iff s is a strict subterm of t. We call $\prec$ the subterm ordering without causing confusion. This is clearly stable. This is the ordering that we are going to use throughout this paper. In particular this means that if a clause contains literals $\pm P(x)$ and $\pm' Q(t)$ where t is non-trivial and contains x, then we cannot choose the literal $\pm P(x)$ to resolve upon in this clause. Because of the simple form of unifiers of irreducible terms we have:

LEMMA 5.4. *Binary ordered resolution and ordered factorization, w.r.t. the subterm ordering, on clauses in $S^\dagger$ produces clauses which are again in $S^\dagger$ (upto renaming).*

PROOF. Factorization on a ground clause does not produce any new clause. Now suppose we factorize the non-ground clause $C[\mathbf{x}_1] \vee \pm P(s[\mathbf{x}_1]) \vee \pm P(t[\mathbf{x}_1])$ to produce the clause $C[\mathbf{x}_1]\sigma \vee \pm P(s[\mathbf{x}_1])\sigma$ where $\sigma = mgu(s[\mathbf{x}_1], t[\mathbf{x}_1])$. If the premise has only trivial literals then factorization is equivalent to doing nothing. Otherwise by ordering constraints, s and t are non-trivial. By Lemma 5.2 either $s[\mathbf{x}_1] = t[\mathbf{x}_1]$ in which case factorization does nothing, or $\mathbf{x}_1\sigma$ is a ground subterm of $s[\mathbf{x}_1]$ or of $t[\mathbf{x}_1]$. In the latter case all literals in $(C[\mathbf{x}_1] \vee P(s[\mathbf{x}_1])\sigma$ are of the form $\pm' Q(t'[\mathbf{x}_1]\sigma))$ where $t'[\mathbf{x}_1] \in \mathsf{Ng}$ and $\mathbf{x}_1\sigma \in \mathsf{G} \subseteq \mathsf{Ngs}[\mathsf{G}]$.

Now we consider binary resolution steps. We have the following cases:

(1) If both clauses are ground then the result is clear.

(2) Now consider both clauses $C_1[\mathbf{x}_1]$ and $C_2[\mathbf{x}_1]$ to be non-ground. Before resolution we rename the second clause to obtain $C_2[\mathbf{x}_2]$. Clearly all literals in $C_1[\mathbf{x}_1]$ and $C_2[\mathbf{x}_1]$ are of the form $\pm Q(u[\mathbf{x}_1])$ where $u[\mathbf{x}_1] \in \mathsf{Ng}$. Let $C_1[\mathbf{x}_1] = C_1'[\mathbf{x}_1] \vee P(s[\mathbf{x}_1])$ and $C_2[\mathbf{x}_2] = -P(t[\mathbf{x}_2]) \vee C_2'[\mathbf{x}_2]$ where $P(s[\mathbf{x}_1])$ and $-P(t[\mathbf{x}_2])$ are the literals to be resolved upon in the respective clauses. If $s[\mathbf{x}_1]$ and $t[\mathbf{x}_2]$ are unifiable then from Lemma 5.1, one of the following cases hold:

—$s[\mathbf{x}_1] = \mathbf{x}_1$ (the case where $t[\mathbf{x}_2] = \mathbf{x}_2$ is treated similarly). From the definition of $\prec$, for $P(s[\mathbf{x}_1])$ to be chosen for resolution, all literals in $C_1'[\mathbf{x}_1]$ are of the form $\pm Q(\mathbf{x}_1)$. The resolvent is $C[\mathbf{x}_2] = C_1'[\mathbf{x}_1]\sigma \cup C_2'$, where $\sigma = \{\mathbf{x}_1 \mapsto t[\mathbf{x}_2]\}$. Each literal in $C_1'[\mathbf{x}_1]\sigma$ is of the form $\pm Q(t[\mathbf{x}_2])$ and each literal in $C_2'[\mathbf{x}_2]$ is of the form $\pm Q(t'[\mathbf{x}_2])$ where $t' \in \mathsf{Ng}$. Hence $C[\mathbf{x}_1] \in S^\dagger$.

—$s[\mathbf{x}_1] = t[\mathbf{x}_1]$. Then the resolvent is $C_1'[\mathbf{x}_1] \vee C_2'[\mathbf{x}_1]$.

—$s[\mathbf{x}_1]$ and $t[\mathbf{x}_2]$ have a mgu σ such that $\mathbf{x}_1\sigma, \mathbf{x}_2\sigma \in \mathsf{Ngs}[\mathsf{G}]$. The resolvent $C_1'[\mathbf{x}_1]\sigma \vee C_2'[\mathbf{x}_2]\sigma$ has only ground atoms of the form $\pm Q(t')$ where $t' \in \mathsf{Ng}[\mathsf{Ngs}[\mathsf{G}]]$.

(3) Now let the first clause $C_1[\mathbf{x}_1] = C_1'[\mathbf{x}_1] \vee \pm P(t[\mathbf{x}_1])$ be non-ground, and the second clause $C_2 = \mp P(s) \vee C_2'$ be ground with $\pm P(t[\mathbf{x}_1])$ and $\mp P(s)$ being the respective literals chosen from $C_1[\mathbf{x}_1]$ and C_2 for resolution. All literals in $C_1[\mathbf{x}_1]$ are of the form $\pm' Q(t'[\mathbf{x}_1])$ with $t' \in \mathsf{Ng}$. All literals in C_2 are of the form $\pm' Q(t')$ with $t' \in \mathsf{Ng}[\mathsf{Ngs}[\mathsf{G}]]$. Suppose that s and $t[\mathbf{x}_1]$ do unify. We have the following cases:

—$s \in \mathsf{Ngs}[\mathsf{G}]$. Then the resolvent $C = C_1'[\mathbf{x}_1]\sigma \cup C_2'$ where $\sigma = \{\mathbf{x}_1 \mapsto g\}$ where g is subterm of s. As $s \in \mathsf{Ngs}[\mathsf{G}]$ hence $g \in \mathsf{Ngs}[\mathsf{G}]$. Hence all literals in $C_1'[\mathbf{x}_1]\sigma$ are of the form $\pm Q(t')$ where $t' \in \mathsf{Ng}[\mathsf{Ngs}[\mathsf{G}]]$. Hence $C \in S^\dagger$.

—Now suppose $s \in \mathsf{Ng}[\mathsf{Ngs}[\mathsf{G}]] \setminus \mathsf{Ngs}[\mathsf{G}]$. We must have $s = s_1[s_2]$ for some non-trivial $s_1[\mathbf{x}_1] \in \mathsf{Ng}$ and some $s_2 \in \mathsf{Ngs}[\mathsf{G}]$. This is the interesting case which shows

why the terms remain in the required form during resolution. The resolvent is $C = C_1'[\mathbf{x}_1]\sigma \vee C_2'$ where $\sigma = \{\mathbf{x}_1 \mapsto g\}$ is the mgu of $t[\mathbf{x}_1]$ and s for some ground term g. As $t[g] = s_1[s_2]$, $\sigma_1 = \{\mathbf{x}_1 \mapsto g, \mathbf{x}_2 \mapsto s_2\}$ is a unifier of the terms $t[\mathbf{x}_1]$ and $s_1[\mathbf{x}_2]$. By Lemma 5.1 we have the following cases:

—$t[\mathbf{x}_1] = \mathbf{x}_1$, so that $g = s \in \mathsf{Ng}[\mathsf{Ngs}[\mathsf{G}]]$. By definition of $\prec$, for $\pm P(t[\mathbf{x}_1])$ to be chosen for resolution, all literals in $C_1[\mathbf{x}_1]$ must be of the form $\pm' Q(\mathbf{x}_1)$. Hence all literals in $C_1'\sigma$ are of the form $\pm' Q(g)$. Hence $C \in S^\dagger$.

—$t[\mathbf{x}_1] = s_1[\mathbf{x}_1]$. Then $g = s_2 \in \mathsf{Ngs}[\mathsf{G}]$. Hence all literals in $C_1'\sigma$ are of the form $\pm' Q(t'[g])$ where $t'[\mathbf{x}_1] \in \mathsf{Ng}$. Hence $C \in S^\dagger$.

—$g = \mathbf{x}_1\sigma \in \mathsf{Ngs}[\mathsf{G}]$. Hence all literals in $C_1'\sigma$ are of the form $\pm' Q(t'[g])$ where $t' \in \mathsf{Ng}$. Hence $C \in S^\dagger$.

□

Hence to decide satisfiability of $S \subseteq S^\dagger$, we keep generating new clauses of $S^\dagger$ by doing ordered binary resolution and ordered factorization w.r.t. the subterm ordering till no new clause can be generated, and then check whether the empty clause has been produced. Also recall that APDS consist of Horn one-variable clauses. Hence:

THEOREM 5.5. *Satisfiability for the classes $\mathcal{V}_1$ and $\mathcal{V}_1 Horn$ is DEXPTIME-complete.*

6. FLAT CLAUSES: RESOLUTION MODULO PROPOSITIONAL REASONING

Next we show how to decide the class $\mathcal{F}$ of flat clauses in NEXPTIME. This is well known when the maximal arity r is a constant, or when all non-trivial literals in a clause have the same *sequence* (instead of the same *set*) of variables. But we are not aware of a proof of the NEXPTIME upper bound in the general case. We show how to obtain NEXPTIME upper bound in the general case, by doing resolution modulo propositional reasoning. While this constitutes an interesting result of its own, the techniques allow us to deal with the full class $\mathcal{C}$ efficiently. Also this shows that the generality of the class $\mathcal{C}$ does not cost more in terms of complexity. An *ϵ-block* is a one-variable clause which contains only trivial literals. A complex clause C is a flat clause $\bigvee_{i=1}^{k} \pm_i P_i(f_i(x_1^i, \ldots, x_{n_i}^i)) \vee \bigvee_{j=1}^{l} \pm_j Q_j(x_j)$ in which $k \geq 1$. Hence a flat clause is either a complex clause, or an *ϵ-clause* which is defined to be a disjunction of ϵ-blocks, i.e. to be of the form $B_1[x_1] \sqcup \ldots \sqcup B_n[x_n]$ where each B_i is an ϵ-block. ϵ-clauses are difficult to deal with, hence we split them to produce ϵ-blocks. Hence define *ϵ-splitting* as the restriction of the splitting rule in which one of the components is an ϵ-block.

Recall that r is the maximal arity of symbols in Σ. Upto renaming, any complex clause C is such that $\mathsf{fv}(C) \subseteq \mathbf{X}_r = \{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$, and any ϵ-block C is such that $\mathsf{fv}(C) \subseteq \{\mathbf{x}_{r+1}\}$. The choice of $\mathbf{x}_{r+1}$ is not crucial. Now notice that ordered resolution between sets of complex clauses and ϵ-blocks only produces flat clauses, which can then be split to be left with only complex clauses and ϵ-blocks. E.g. Resolution between

$$P_1(\mathbf{x}_1) \vee -P_2(\mathbf{x}_2) \vee P_3(f(\mathbf{x}_1, \mathbf{x}_2)) \vee -P_4(g(\mathbf{x}_2, \mathbf{x}_1))$$

and

$$P_4(g(\mathbf{x}_1, \mathbf{x}_1)) \vee -P_5(h(\mathbf{x}_1)) \vee P_6(\mathbf{x}_1)$$

produces

$$P_1(\mathbf{x}_1) \vee -P_2(\mathbf{x}_1) \vee P_3(f(\mathbf{x}_1, \mathbf{x}_1)) \vee -P_5(h(\mathbf{x}_1)) \vee P_6(\mathbf{x}_1)$$

Resolution between

$$P_2(\mathbf{x}_{r+1}) \quad \text{and} \quad -P_2(f(\mathbf{x}_1,\mathbf{x}_2)) \vee P_3(\mathbf{x}_1) \vee P_4(\mathbf{x}_2)$$

produces $P_3(\mathbf{x}_1) \vee P_4(\mathbf{x}_2)$ which can then be split. The point is that we always choose a non-trivial literal from a clause for resolution, if there is one. Complex clauses obtained after resolution and splitting can be renamed to contain variables only from $\mathbf{X}_r$. As there are finitely many complex clauses and ϵ-blocks this gives us a decision procedure. Note however that the number of complex clauses is doubly exponential. This is because we allow clauses of the form $P_1(f_1(\mathbf{x}_1,\mathbf{x}_1,\mathbf{x}_2)) \vee P_2(f_2(\mathbf{x}_2,\mathbf{x}_1)) \vee P_3(f_3(\mathbf{x}_2,\mathbf{x}_1,\mathbf{x}_2)) \vee ...$, i.e. the nontrivial terms contain arbitrary number of repetitions of variables in arbitrary order. The number of such variable sequences of r variables is exponentially many, hence the number of clauses is doubly exponential. Letting the maximal arity r to be a constant, or forcing all non-trivial literals in a clause to have the same variable sequence would have produced only exponentially many clauses. In presence of splitting, this would have given us the well-known NEXPTIME upper bound, which is also optimal. But we are not aware of a proof of the NEXPTIME upper bound in the general case. To obtain NEXPTIME upper bound in the general case we introduce the technique of resolution modulo propositional reasoning.

Definition 6.1. For a clause C, define the set of its projections as $\pi(C) = C[\mathbf{X}_r]$. Define the set $\mathsf{U} = \{f(x_1,\ldots,x_n) \mid f \in \Sigma \text{ and each } x_i \in \mathbf{X}_r\}$ of size exponential in r.

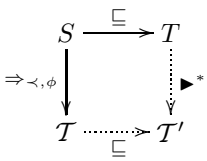
Essentially projection involves making certain variables in a clause equal. Our intuition is that resolution between two complex clauses amounts to propositional resolution between their projections. Resolution between an ϵ-block C_1 and a complex clause C_2, which has variables only from $\mathbf{X}_r$, amounts to propositional resolution of a clause from $C_1[\mathsf{U}]$ with C_2. Also note that propositional resolution followed by further projection is equivalent to projection followed by propositional resolution. Each complex clause has exponentially many projections. This suggests that we can compute beforehand the exponentially many projections of complex clauses and exponentially many instantiations of ϵ-blocks. All new complex clauses generated by propositional resolution are ignored. But after several such propositional resolution steps, we may get an ϵ-clause, which should then be split and instantiated and used for obtaining further propositional resolvents. In other words we only compute such propositionally implied ϵ-clauses, do splitting and instantiation and iterate the process. This generates all resolvents upto propositional implication. We now formalize our approach. We start with the following observation which is used in this and further sections.

LEMMA 6.2. *Let $x_1,\ldots,x_n,y_1,\ldots,y_n$ be variables, not necessarily distinct, but with $\{x_1,\ldots,x_n\} \cap \{y_1,\ldots,y_n\} = \emptyset$. Then the terms $f(x_1,\ldots,x_n)$ and $f(y_1,\ldots,y_n)$ have an mgu σ such that $\{x_1,\ldots,x_n\}\sigma \subseteq \{x_1,\ldots,x_n\}$ and $y_i\sigma = x_i\sigma$ for $1 \leq i \leq n$.*

Definition 6.3. In the rest of this section, for a set S of clauses, $\mathsf{comp}(S)$ is the set of complex clauses in S, $\mathsf{eps}(S)$ the set of ϵ-blocks in S, $\pi(S) = \bigcup_{C\in S} \pi(C)$ and $\mathsf{I}(S) = \pi(\mathsf{comp}(S)) \cup \mathsf{eps}(S)[\mathbf{x}_{r+1}] \cup \mathsf{eps}(S)[\mathsf{U}]$. For sets S and T of complex clauses and ϵ-blocks, $S \sqsubseteq T$ means that:
– if $C \in S$ is a complex clause then $\mathsf{I}(T) \models_{\mathrm{p}} \pi(C)$, and
– if $C \in S$ is an ϵ-block then $C[\mathbf{x}_{r+1}] \in \mathsf{eps}(T)[\mathbf{x}_{r+1}]$.

For tableaux $\mathcal{T}_1$ and $\mathcal{T}_2$ involving only complex clauses and ϵ-blocks we write $\mathcal{T}_1 \sqsubseteq \mathcal{T}_2$ if $\mathcal{T}_1$ can be written as $S_1 \mid \ldots \mid S_n$ and $\mathcal{T}_2$ can be written as $T_1 \mid \ldots \mid T_n$ (note same n) such that $S_i \sqsubseteq T_i$ for $1 \leq i \leq n$. Intuitively $\mathcal{T}_2$ is a succinct representation of $\mathcal{T}_1$. Instead of performing actual resolution and splitting steps, we will simulate them by new rules which work on the succinct representations. Define the splitting strategy ϕ as the one which repeatedly applies ϵ-splitting on a tableau as long as possible. The relation $\Rightarrow_{\prec,\phi}$ provides us a sound and complete method for testing unsatisfiability. We define the alternative procedure for testing unsatisfiability by using succinct representations of tableaux. We define $\blacktriangleright$ by the rule: $\mathcal{T} \mid S \blacktriangleright \mathcal{T} \mid S \cup \{B_1\} \mid \ldots \mid S \cup \{B_k\}$ whenever $\mathsf{I}(S) \models_{\mathrm{p}} C = B_1[\mathbf{x}_{i_1}] \sqcup \ldots \sqcup B_k[\mathbf{x}_{i_k}]$, C is an ϵ-clause, and $1 \leq i_1, \ldots, i_k \leq r+1$. Then $\blacktriangleright$ simulates $\Rightarrow_{\prec,\phi}$:

LEMMA 6.4. *If S is a set of complex clauses and ϵ-blocks, $S \sqsubseteq T$ and $S \Rightarrow_{\prec,\phi} \mathcal{T}$, then all clauses occurring in $\mathcal{T}$ are complex clauses or ϵ-blocks and $T \blacktriangleright^* \mathcal{T}'$ for some $\mathcal{T}'$ such that $\mathcal{T} \sqsubseteq \mathcal{T}'$.*



PROOF. We have the following ways in which $\mathcal{T}$ is obtained from S by doing one resolution step followed by splitting:

(1) We resolve two ϵ-blocks C_1 and C_2 of S to get an ϵ-block C, and $\mathcal{T} = S \cup \{C\}$. Then $\{C_1[\mathbf{x}_{r+1}], C_2[\mathbf{x}_{r+1}]\} \models_{\mathrm{p}} C[\mathbf{x}_{r+1}]$. Also as $S \sqsubseteq T$ we have $\{C_1[\mathbf{x}_{r+1}], C_2[\mathbf{x}_{r+1}]\} \subseteq \mathsf{eps}(T)[\mathbf{x}_{r+1}]$. We have $\mathsf{I}(T) \models_{\mathrm{p}} C[\mathbf{x}_{r+1}]$. Hence $T \blacktriangleright T \cup \{C[\mathbf{x}_{r+1}]\}$ and clearly $S \cup \{C\} \sqsubseteq T \cup \{C\}$.

(2) We resolve an ϵ-block $C_1[\mathbf{x}_{r+1}]$ with a complex clause $C_2[\mathbf{x}_1, \ldots, \mathbf{x}_r]$, both from S upto renaming, and we have $C_1[\mathbf{x}_{r+1}] \in \mathsf{eps}(T)[\mathbf{x}_{r+1}]$ and $\mathsf{I}(T) \models_{\mathrm{p}} \pi(C_2)$. By ordering constraints, we have $C_1[\mathbf{x}_{r+1}] = C_1'[\mathbf{x}_{r+1}] \vee \pm P(\mathbf{x}_{r+1})$ and $C_2[\mathbf{x}_1, \ldots, \mathbf{x}_r] = \mp P(f(x_1, \ldots, x_n)) \vee C_2'[\mathbf{x}_1, \ldots, \mathbf{x}_r]$ so that resolution produces $C[\mathbf{x}_1, \ldots, \mathbf{x}_r] = C_1'[f(x_1, \ldots, x_n)] \vee C_2'[\mathbf{x}_1, \ldots, \mathbf{x}_r]$. Clearly $C_1[\mathsf{U}] \cup \{C_2[\mathbf{x}_1, \ldots, \mathbf{x}_r]\} \models_{\mathrm{p}} C[\mathbf{x}_1, \ldots, \mathbf{x}_r]$. Also $\pi(C_1[\mathsf{U}]) = C_1[\mathsf{U}]$. Hence $\mathsf{I}(T) \models_{\mathrm{p}} C_1[\mathsf{U}] \cup \pi(C_2) \models_{\mathrm{p}} \pi(C) \supseteq \{C[\mathbf{x}_1, \ldots, \mathbf{x}_r]\}$.

—If C_1' is not empty or if C_2' has some non-trivial literal then C is a complex clause and $\mathcal{T} = S \cup \{C\} \sqsubseteq T$.

—If C_1' is empty and C_2' has only trivial literals then $C[\mathbf{x}_1, \ldots, \mathbf{x}_r]$ is an ϵ-clause of the form $B_1[\mathbf{x}_{i_1}] \sqcup \ldots \sqcup B_k[\mathbf{x}_{i_k}]$ with $1 \leq i_1, \ldots, i_k \leq r$. $\mathcal{T} = S \cup \{B_1\} \mid \ldots \mid S \cup \{B_k\}$. Since $\mathsf{I}(T) \models_{\mathrm{p}} C[\mathbf{x}_1, \ldots, \mathbf{x}_r]$, hence $T \blacktriangleright \mathcal{T}'$ where $\mathcal{T}' = T \cup \{B_1\} \mid \ldots \mid T \cup \{B_k\}$ and we have $\mathcal{T} \sqsubseteq \mathcal{T}'$.

(3) We resolve two complex clauses $C_1[\mathbf{x}_1, \ldots, \mathbf{x}_r]$ and $C_2[\mathbf{x}_1, \ldots, \mathbf{x}_r]$, both from S upto renaming, and we have $\mathsf{I}(T) \models_{\mathrm{p}} \pi(C_1)$ and $\mathsf{I}(T) \models_{\mathrm{p}} \pi(C_2)$. First we rename the second clause as $C_2[\mathbf{x}_{r+1}, \ldots, \mathbf{x}_{2r}]$ by applying the renaming $\sigma_0 = \{\mathbf{x}_1 \mapsto \mathbf{x}_{r+1}, \ldots, \mathbf{x}_r \mapsto \mathbf{x}_{2r}\}$. By ordering constraints, $C_1[\mathbf{x}_1, \ldots, \mathbf{x}_r]$ is of the form $C_1'[\mathbf{x}_1, \ldots, \mathbf{x}_r] \vee \pm P(f(x_1, \ldots, x_n))$ and $C_2[\mathbf{x}_{r+1}, \ldots, \mathbf{x}_{2r}]$ is of the form $\mp P(f(y_1, \ldots, y_n)) \vee C_2'[\mathbf{x}_{r+1}, \ldots, \mathbf{x}_{2r}]$ so that $\pm P(f(x_1, \ldots, x_n))$ and $\mp P(f(y_1, \ldots, y_n))$ are the literals to be resolved from the respective clauses. By Lemma 6.2, the resolvent is $C = C_1'[\mathbf{x}_1, \ldots, \mathbf{x}_r]\sigma \vee C_2'[\mathbf{x}_{r+1}, \ldots, \mathbf{x}_{2r}]\sigma$ where σ is such that $\{x_1, \ldots, x_n\}\sigma \subseteq \{x_1, \ldots, x_n\}$ and $y_i\sigma = x_i\sigma$ for

$1 \leq i \leq n$. C is obtained by propositional resolution from $C_1[\mathbf{x}_1, \ldots, \mathbf{x}_r]\sigma \in \pi(C_1)$ and $C_2[\mathbf{x}_{r+1}, \ldots, \mathbf{x}_{2r}]\sigma = C_2[\mathbf{x}_1, \ldots, \mathbf{x}_r]\sigma_0\sigma \in \pi(C_2)$. Hence $\pi(C_1) \cup \pi(C_2) \models_{\mathrm{p}} C[\mathbf{x}_1, \ldots, \mathbf{x}_r]$. Hence $\pi(\pi(C_1)) \cup \pi(\pi(C_2)) = \pi(C_1) \cup \pi(C_2) \models_{\mathrm{p}} \pi(C)$. As $\mathsf{I}(T) \models_{\mathrm{p}} \pi(C_1)$ and $\mathsf{I}(T) \models_{\mathrm{p}} \pi(C_2)$. Hence $\mathsf{I}(T) \models_{\mathrm{p}} \pi(C) \supseteq \{C[\mathbf{x}_1, \ldots, \mathbf{x}_r])\}$.

—If either C_1' or C_2' contains a non-trivial literal then C is a complex clause and $\mathcal{T} = S \cup \{C\} \sqsubseteq T$.

—If C_1' and C_2' contain only trivial literals then $C[\mathbf{x}_1, \ldots, \mathbf{x}_r]$ is an ϵ-clause of the form $B_1[\mathbf{x}_{i_1}] \sqcup \ldots \sqcup B_k[\mathbf{x}_{i_k}]$ with $1 \leq i_1, \ldots, i_k \leq r$. $\mathcal{T} = S \cup \{B_1\} \mid \ldots \mid S \cup \{B_k\}$. As $\mathsf{I}(T) \models_{\mathrm{p}} C[\mathbf{x}_1, \ldots, \mathbf{x}_r]$ we have $T \blacktriangleright \mathcal{T}'$ where $\mathcal{T}' = T \cup \{B_1\} \mid \ldots \mid T \cup \{B_k\}$. Also $\mathcal{T} \sqsubseteq \mathcal{T}'$.

(4) $C[\mathbf{x}_1, \ldots, \mathbf{x}_r]$ is a renaming of a complex clause in S, and we factor $C[\mathbf{x}_1, \ldots, \mathbf{x}_r]$ to get a complex clause $C[\mathbf{x}_1, \ldots, \mathbf{x}_r]\sigma$ where $\mathbf{X}_r\sigma \subseteq \mathbf{X}_r$, and $\mathcal{T} = S \cup \{C[\mathbf{x}_1, \ldots, \mathbf{x}_r]\sigma\}$. $C[\mathbf{x}_1, \ldots, \mathbf{x}_r]\sigma \in \pi(C)$. Hence $\pi(\{C[\mathbf{x}_1, \ldots, \mathbf{x}_r]\sigma\}) \subseteq \pi(\pi(C)) = \pi(C)$. As $S \sqsubseteq T$ hence $\mathsf{I}(T) \models_{\mathrm{p}} \pi(C)$. Hence $\mathsf{I}(T) \models_{\mathrm{p}} \pi(\{C[\mathbf{x}_1, \ldots, \mathbf{x}_r]\sigma\})$. Hence we have $\mathcal{T} = S \cup \{C[\mathbf{x}_1, \ldots, \mathbf{x}_r]\sigma\} \sqsubseteq T$.

□

Hence we have completeness of $\blacktriangleright$:

LEMMA 6.5. *If a set S of complex clauses and ϵ-blocks is unsatisfiable then $S \blacktriangleright^* \mathcal{T}$ for some closed $\mathcal{T}$.*

PROOF. By Lemma 3.1, $S \Rightarrow^*_{\prec,\phi} S_1 \mid \ldots \mid S_n$ such that each S_i contains the empty clause $\Box$. As $S \sqsubseteq S$, hence by Lemma 6.4, we have some $T_1, \ldots, T_n$ such that $S \blacktriangleright^* T_1 \mid \ldots \mid T_n$ and $S_i \sqsubseteq T_i$ for $1 \leq i \leq n$. Since $\Box \in S_i$ and $\Box$ is an ϵ-block, hence $\Box \in T_i$ for $1 \leq i \leq n$. □

Call a set S of complex clauses and ϵ-blocks *saturated* if the following condition is satisfied: if $\mathsf{I}(S) \models_{\mathrm{p}} B_1[\mathbf{x}_{i_1}] \sqcup \ldots \sqcup B_k[\mathbf{x}_{i_k}]$ with $1 \leq i_1, \ldots, i_k \leq r+1$, each B_i being an ϵ-block, then there is some $1 \leq j \leq k$ such that $B_j[\mathbf{x}_{r+1}] \in S[\mathbf{x}_{r+1}]$.

LEMMA 6.6. *If S is a satisfiable set of complex clauses and ϵ-blocks then $S \blacktriangleright^* \mathcal{T} \mid T$ for some $\mathcal{T}$ and some saturated set T of complex clauses and ϵ-blocks, such that $\Box \notin T$.*

PROOF. We construct a sequence $S = S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots$ of complex clauses and ϵ-blocks such that S_i is satisfiable and $S_i \blacktriangleright^* S_{i+1} \mid \mathcal{T}_i$ for some $\mathcal{T}_i$ for each i. $S = S_0$ is satisfiable by assumption. Now assume we have already defined $S_0, \ldots, S_i$ and $\mathcal{T}_0, \ldots, \mathcal{T}_{i-1}$. Let $C^\ell = B_1^\ell[\mathbf{x}_{i_1^\ell}] \sqcup \ldots \sqcup B_{k_\ell}^\ell[\mathbf{x}_{i_{k_\ell}^\ell}]$ for $1 \leq \ell \leq N$ be all the possible ϵ-clauses such that $\mathsf{I}(S_i) \models_{\mathrm{p}} C^\ell$, $1 \leq i_1^\ell, \ldots, i_{k_\ell}^\ell \leq r+1$. Since S_i is satisfiable, $S_i \cup \{C^\ell \mid 1 \leq \ell \leq N\}$ is satisfiable. Since $\mathbf{x}_{i_1^\ell}, \ldots, \mathbf{x}_{i_{k_\ell}^\ell}$ are mutually distinct for $1 \leq \ell \leq N$, there are $1 \leq j_\ell \leq k_\ell$ for $1 \leq \ell \leq N$ such that $S_i \cup \{B_{j_\ell}^\ell \mid 1 \leq \ell \leq N\}$ is satisfiable. Let $S_{i+1} = S_i \cup \{B_{j_\ell}^\ell \mid 1 \leq \ell \leq N\}$. S_{i+1} is satisfiable. Also it is clear that $S_i \blacktriangleright^* S_{i+1} \mid \mathcal{T}_i$ for some $\mathcal{T}_i$. If $S_{i+1} = S_i$ then S_i is saturated, otherwise S_{i+1} has strictly more ϵ-blocks upto renaming. As there are only finitely many ϵ-blocks upto renaming, eventually we will end up with a saturated set T in this way. Since T is satisfiable, $\Box \notin T$. From construction it is clear that there is some $\mathcal{T}$ such that $S \blacktriangleright^* \mathcal{T} \mid T$. □

THEOREM 6.7. *Satisfiability for the class $\mathcal{F}$ is NEXPTIME-complete.*

PROOF. The lower bound comes from reduction of satisfiability of positive set constraints which is NEXPTIME-complete [Aiken et al. 1993]. For the upper bound let S be a finite set of flat clauses. Repeatedly apply ϵ-splitting to obtain $f(S) = S_1 \mid \ldots \mid S_m$. S is satisfiable iff some S_i is satisfiable. The number m of branches in $f(S)$ is at most exponential. Also each branch has size linear in the size of S. We non-deterministically choose some S_i and check its satisfiability in NEXPTIME.

Hence wlog we may assume that the given set S has only complex clauses and ϵ-blocks. We non-deterministically choose a certain number of ϵ-blocks $B_1, \ldots, B_N$ and check that $T = S \cup \{B_1, \ldots, B_N\}$ is saturated and $\Box \notin T$. By Lemma 6.6, if S is satisfiable then clearly there is such a set T. Conversely if there is such a set T, then since T is saturated, whenever $T \blacktriangleright^* \mathcal{T}$, we will have $\mathcal{T} = T \mid \mathcal{T}'$ for some $\mathcal{T}'$. Hence we can never have $T \blacktriangleright^* \mathcal{T}$ where $\mathcal{T}$ is closed. Then by Lemma 6.5 we conclude that T is satisfiable. Hence $S \subseteq T$ is also satisfiable.

Guessing the set T requires non-deterministically choosing from among exponentially many ϵ-blocks. To check that T is saturated, for every ϵ-clause $C = B_1[\mathbf{x}_{i_1}] \sqcup \ldots \sqcup B_k[\mathbf{x}_{i_k}]$, with $1 \leq i_1, \ldots, i_k \leq r+1$, and $B_j[\mathbf{x}_{r+1}] \notin T[\mathbf{x}_{r+1}]$ for $1 \leq j \leq k$, we check that $\mathsf{I}(T) \not\models_{\mathrm{p}} C$, i.e. $\mathsf{I}(T) \cup \neg C$ is propositionally satisfiable (where $\neg(L_1 \vee \ldots \vee L_n)$ denotes $\{-L_1, \ldots, -L_n\}$). This can be checked in NEXPTIME since propositional satisfiability can be checked in NP. We need to do such checks for at most exponentially many possible values of C. □

7. COMBINATION: ORDERED LITERAL REPLACEMENT

Combining flat and one-variable clauses creates additional difficulties. First observe that resolving a one variable clause $C_1 \vee \pm P(f(s_1[x], \ldots, s_n[x]))$ with a complex clause $\mp P(f(x_1, \ldots, x_n)) \vee C_2$ produces a one-variable clause. If $s_i[x] = s_j[x]$ for all $x_i = x_j$, and if C_2 contains a literal $P(x_i)$ then the resolvent contains a literal $P(s_i[x])$. The problem now is that even if $f(s_1[x], \ldots, s_n[x])$ is irreducible, $s_i[x]$ may not be irreducible. E.g. $f(g(h(x)), x)$ is irreducible but $g(h(x))$ is not irreducible. As in Section 5 we may think of replacing this literal by simpler literals involving fresh predicates. Firstly we have to ensure that in this process we do not generate infinitely many predicates. Secondly it is not clear that mixing ordered resolution steps with replacement of literals is still complete. Correctness is easy to show since the new clause is in some sense equivalent to the old deleted clause. However deletion of clauses arbitrarily can violate completeness of the resolution procedure. The key factor which preserves completeness is that we replace literals by smaller literals w.r.t. the given ordering $<$.

Formally a *replacement rule* is of the form $A_1 \to A_2$ where A_1 and A_2 are (not necessarily ground) atoms. The clause set *associated* with this rule is $\{A_1 \vee -A_2, -A_1 \vee A_2\}$. Intuitively such a replacement rule says that A_1 and A_2 are equivalent. The clause set $cl(\mathcal{R})$ associated with a set $\mathcal{R}$ of replacement rules is the union of the clause sets associated with the individual replacement rules in $\mathcal{R}$. Given a stable ordering $<$ on atoms, a replacement rule $A_1 \to A_2$ is *ordered* iff $A_2 < A_1$. We define the relation $\to_{\mathcal{R}}$ as: $S \to_{\mathcal{R}} (S \setminus \{\pm A_1\sigma \vee C\}) \cup \{\pm A_2\sigma \vee C\}$ whenever S is a set of clauses, $\pm A_1\sigma \vee C \in S$, $A_1 \to A_2 \in \mathcal{R}$ and σ is some substitution. Hence we replace literals in a clause by smaller literals. The relation is extended to tableaux as usual.

Next note that in the above resolution example, even if $f(s_1[x], \ldots, s_n[x])$ is non-ground, some s_i may be ground. Hence the resolvent may have ground as well as non-

ground literals. We avoided this in Section 5 by initial preprocessing. Now we may think of splitting these resolvents during the resolution procedure. This however will be difficult to simulate using the alternative resolution procedure on succinct representations of tableaux because we will generate doubly exponentially many one-variable clauses. To avoid this we use a variant of splitting called *splitting-with-naming* [Riazanov and Voronkov 2001]. Instead of creating two branches after splitting, this rule puts both components into the same set, but with tags to simulate branches produced by ordinary splitting. Fix a finite set $\mathbb{P}$ of predicate symbols. $\mathbb{P}$-clauses are clauses whose predicates are all from $\mathbb{P}$. Introduce fresh zero-ary predicates $\overline{C}$ for $\mathbb{P}$-clauses C modulo renaming, i.e. $\overline{C_1} = \overline{C_2}$ iff $C_1\sigma = C_2$ for some renaming σ. Literals $\pm\overline{C}$ for $\mathbb{P}$-clauses C are *splitting literals*. The *splitting-with-naming* rule is defined as: $S \rightarrow_{nspl} (S \setminus \{C_1 \sqcup C_2\}) \cup \{C_1 \vee -\overline{C_2}, \overline{C_2} \vee C_2\}$ where $C_1 \sqcup C_2 \in S$, C_2 is non-empty and has only non-splitting literals, and C_1 has at least one non-splitting literal. Intuitively $\overline{C_2}$ represents the negation of C_2. We will use both splitting and splitting-with-naming according to some predefined strategy. Hence for a finite set $\mathcal{Q}$ of splitting atoms, define *$\mathcal{Q}$-splitting* as the restriction of the splitting-with-naming rule where the splitting atom produced is restricted to be from $\mathcal{Q}$. Call this restricted relation as $\rightarrow_{\mathcal{Q}-nspl}$. This is extended to tableaux as usual. Now once we have generated the clauses $C_1 \vee -\overline{C_2}$ and $\overline{C_2} \vee C_2$ we would like to keep resolving on the second part of the second clause till we are left with the clause $\overline{C_2}$ (possibly with other positive splitting literals) which would then be resolved with the first clause to produce C_1 (possibly with other positive splitting literals) and only then the literals in C_1 would be resolved upon. Such a strategy cannot be ensured by ordered resolution, hence we introduce a new rule. An ordering $<$ over non-splitting atoms is extended to the ordering $<_s$ by letting $q <_s A$ whenever q is a splitting atom and A is a non-splitting atom, and $A <_s B$ whenever A, B are non-splitting atoms and $A < B$. We define *modified ordered binary resolution* by the following rule:

$$\frac{C_1 \vee A \quad -B \vee C_2}{C_1\sigma \vee C_2\sigma}$$

where $\sigma = mgu(A, B)$ and the following conditions are satisfied:
(1) C_1 has no negative splitting literal, and A is maximal in C_1.
(2) (a) either $B \in \mathcal{Q}$, or
(b) C_2 has no negative splitting literal, and B is maximal in C_2.
As usual we rename the premises before resolution so that they do not share variables. This rule says that we must select a negative splitting literal to resolve upon in any clause, provided the clause has at least one such literal. If no such literal is present in the clause, then the ordering $<_s$ enforces that a positive splitting literal will not be selected as long as the clause has some non-splitting literal. We write $S \Rrightarrow_{<_s} S \cup \{C\}$ to say that C is obtained by one application of the modified binary ordered resolution or the (unmodified) ordered factorization rule on clauses in S. This is extended to tableaux as usual. A *$\mathcal{Q}$-splitting-replacement strategy* is a function ϕ such that $\mathcal{T}(\rightarrow_{\mathcal{Q}-nspl} \cup \rightarrow_{spl} \cup \rightarrow_{\mathcal{R}})^*\phi(\mathcal{T})$ for any tableaux $\mathcal{T}$. Hence we allow both normal splitting and $\mathcal{Q}$-splitting. Modified ordered resolution with $\mathcal{Q}$-splitting-replacement strategy ϕ is defined by the relation: $S \Rrightarrow_{<_s,\phi,\mathcal{R}} \phi(T)$ whenever $S \cup cl(\mathcal{R}) \Rrightarrow_{<_s} T$. This is extended to tableaux as usual. The above modified ordered binary resolution rule can be considered as an instance of *ordered resolution with selection* [Bachmair and Ganzinger 2001], which is known to be sound and complete even with splitting and its variants. Our manner of extending $<$ to $<_s$ is essential for complete-

ness. We now show that soundness and completeness hold even under arbitrary ordered replacement strategies.[1] Wlog we forbid the useless case of replacement rules containing splitting symbols. The relation $<$ is *enumerable* if the set of all ground atoms can be enumerated as $A_1, A_2, \ldots$ such that if $A_i < A_j$ then $i < j$. The subterm ordering is enumerable.

THEOREM 7.1. *Modified ordered resolution, w.r.t. a stable and enumerable ordering, with splitting and $\mathcal{Q}$-splitting and ordered literal replacement is sound and complete for any strategy. I.e. for any set S of $\mathbb{P}$-clauses, for any strict stable and enumerable partial order $<$ on atoms, for any set $\mathcal{R}$ of ordered replacement rules, for any finite set $\mathcal{Q}$ of splitting atoms, and for any $\mathcal{Q}$-splitting-replacement strategy ϕ, $S \cup cl(\mathcal{R})$ is unsatisfiable iff $S \Rrightarrow^*_{<_s,\phi,\mathcal{R}} \mathcal{T}$ for some closed $\mathcal{T}$.*

PROOF. See Appendix B. □

For the rest of this section fix a set $\mathbb{S}$ of one-variable $\mathbb{P}$-clauses and complex $\mathbb{P}$-clauses whose satisfiability we need to decide.

Definition 7.2. Let Ng be the set of non-ground terms occurring as arguments in literals in the one-variable clauses of $\mathbb{S}$. We rename all terms in Ng to contain only the variable $\mathbf{x}_{r+1}$. Wlog assume $\mathbf{x}_{r+1} \in \mathsf{Ng}$. Let Ngs be the set of non-ground subterms of terms in Ng, and $\mathsf{Ngr} = \{s[\mathbf{x}_{r+1}] \mid s$ is non-ground and irreducible,and for some $t, s[t] \in \mathsf{Ngs}\}$. Define $\mathsf{Ngrr} = \{s_1[\ldots[s_m]\ldots] \mid s_1[\ldots[s_n]\ldots] \in \mathsf{Ngs}, m \leq n,$ and each s_i is non-trivial and irreducible$\}$. Define the set of predicates $\mathbb{Q} = \{Ps \mid P \in \mathbb{P}, s \in \mathsf{Ngrr}\}$. Note that $\mathbb{P} \subseteq \mathbb{Q}$. Define the set of replacement rules $\mathcal{R} = \{Ps_1 \ldots s_{m-1}(s_m[\mathbf{x}_{r+1}]) \rightarrow Ps_1 \ldots s_m([\mathbf{x}_{r+1}]) \mid Ps_1 \ldots s_m \in \mathbb{Q}, s_m$ is non-trivial$\}$. They are clearly ordered w.r.t. the subterm ordering $\prec$ defied in Section 5. Let G be the set of ground subterms of terms occurring as arguments in literals in $\mathbb{S}$. Define the set $\mathcal{Q}_0 = \{\pm\overline{P(t)} \mid P \in \mathbb{P}, t \in \mathsf{G}\}$ of splitting atoms.

The purpose of the splitting atoms in $\mathcal{Q}_0$ is to remove ground literals from a non-ground clause. All sets defined above have polynomial size.

Definition 7.3. Let $\mathcal{Q} \supseteq \mathcal{Q}_0$ be any finite set of splitting atoms.

For dealing with the class $\mathcal{C}$ we only need $\mathcal{Q} = \mathcal{Q}_0$, but for a more precise analysis of the Horn fragment in the next Section, we need $\mathcal{Q}$ to also contain some other splitting atoms.

Definition 7.4. Define the sets $\mathsf{Ngr_1} = \{\mathbf{x}_{r+1}\} \cup \{f(s_1, \ldots, s_n) \mid \exists g(t_1, \ldots, t_m) \in \mathsf{Ngr} \cdot \{s_1, \ldots, s_n\} = \{t_1, \ldots, t_m\}\}$ and $\mathsf{G_1} = \{f(s_1, \ldots, s_n) \mid \exists g(t_1, \ldots, t_m) \in \mathsf{G} \mid \{s_1, \ldots, s_n\} = \{t_1, \ldots, t_m\}\}$.

Both $\mathsf{Ngr_1}$ and $\mathsf{G_1}$ have exponential size. The terms in $\mathsf{Ngr_1}$ are produced by resolution of non-ground one-variable clauses with complex clauses, and are also irreducible. In the ground case we get terms in $\mathsf{G_1}$. We will implicitly use various relationships between these sets.

LEMMA 7.5. (*1*) *Every ground subterm of a term in* Ngr *(resp.* $\mathsf{Ngr_1}$*) is in* G.

[1] It seems that completeness can be obtained as a special case of Bachmair and Ganzinger's general redundancy criteria [Bachmair and Ganzinger 2001], as suggested by an anonymous referee. As we have not checked all the details, we have still kept our original alternative proof.

(*2*) *Every non-ground strict subterm of a term in* Ngr *(resp.* Ngr_1*) is in* Ngrr.

(*3*) *In particular, every strict subterm of a term in* Ngr *(resp.* Ngr_1*) is in* $\mathsf{Ngrr} \cup \mathsf{G}$.

For a set $\mathbb{P}'$ of predicates and a set U of terms, the set $\mathbb{P}'[U]$ of atoms is defined as usual. For a set V of atoms the set $-V$ and $\pm V$ of literals is defined as usual. The following types of clauses will be required during resolution:

(C1) clauses $C \vee D$, where C is an ϵ-block with predicates from $\mathbb{Q}$, and $D \subseteq \pm\mathcal{Q}$.

(C2) clauses $C \vee D$ where C is a renaming of a one-variable clause with literals from $\pm\mathbb{Q}(\mathsf{Ngr}_1)$, C has at least one non-trivial literal, and $D \subseteq \pm\mathcal{Q}$.

(C3) clauses $C \vee D$ where C is a non-empty clause with literals from $\pm\mathbb{Q}(\mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]])$, and $D \subseteq \pm\mathcal{Q}$.

(C4) clauses $C \vee D$ where $C = \bigvee_{i=1}^{k} \pm_i P_i(f_i(x_1^i, \ldots, x_{n_i}^i)) \vee \bigvee_{j=1}^{l} \pm_j Q_j(x_j)$ is a complex clause with each $P_i \in \mathbb{Q}$, each $n_i \geq 2$, each $Q_j \in \mathbb{P}$ and $D \subseteq \pm\mathcal{Q}$.

We have already argued why we need splitting literals in the above clauses, and why we need Ngr_1 instead of Ngr in type C2. In type C3 we have Ngrr in place of the set Ngs that we had in Section 5, to take care of interactions between one-variable clauses and complex clauses. In type C4 the trivial literals involve predicates only from $\mathbb{P}$ (and not $\mathbb{Q}$). This is what ensures that we need only finitely many fresh predicates (those from $\mathbb{Q} \setminus \mathbb{P}$) because these are the literals that are involved in replacements when this clause is resolved with a one-variable clause. We have also required that each $n_i \geq 2$. This is only to ensure that types C2 and C4 are disjoint. The clauses that are excluded because of this condition are necessarily of type C2.

Definition 7.6. The $\mathcal{Q}_0$-splitting steps that we use in this section consist of replacing a tableau $\mathcal{T} \mid S$ by the tableau $\mathcal{T} \mid (S \setminus \{C \vee L\}) \cup \{C \vee -\overline{L}, \overline{L} \vee L\}$, where C is non-ground, $L \in \pm\mathbb{P}(\mathsf{G})$ and $C \vee L \in S$. The replacement steps we are going to use are of the following kind:

(1) replacing clause

$$C_1[x] = C \vee \pm P(t_1[\ldots[t_n[x]]\ldots])$$

by clause

$$C_2[x] = C \vee \pm P t_1 \ldots t_{n-1}(t_n[x])$$

where $P \in \mathbb{P}$, $t_{n-1}, t_n[\mathbf{x}_{r+1}] \in \mathsf{Ngr}$ are non-trivial, and $t_1[\ldots[t_n]\ldots] \in \mathsf{Ngrr}$. We have

$$\{C_1[\mathbf{x}_{r+1}]\} \cup cl(\mathcal{R})[\mathsf{Ngrr}] \vDash_{\mathrm{p}} C_2[\mathbf{x}_{r+1}]$$

(2) replacing ground clause

$$C_1 = C \vee \pm P(t_1[\ldots[t_n[g]]\ldots])$$

by clause

$$C_2 = C \vee \pm P t_1 \ldots t_{n-1}(t_n[g])$$

where $P \in \mathbb{P}, g \in \mathsf{Ngrr}[\mathsf{G}_1]$, $t_1[\ldots[t_n]\ldots] \in \mathsf{Ngrr}$, and $t_{n-1}, t_n \in \mathsf{Ngr}$ are non-trivial. This replacement is done only when $t_1[\ldots[t_n[g]]\ldots] \in \mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G}_1]] \setminus \mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]]$. We have

$$\{C_1\} \cup cl(\mathcal{R})[\mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G}_1]]] \vDash_{\mathrm{p}} C_2$$

Define the $\mathcal{Q}_0$-splitting-replacement strategy ϕ as one which repeatedly applies first ϵ-splitting, then the above $\mathcal{Q}_0$-splitting steps, then the above two replacement steps till no further change is possible. Then $\Rrightarrow_{\prec_s,\phi,\mathcal{R}}$ gives us a sound and complete method for testing unsatisfiability.

As in Section 6 we now define a succinct representation of tableaux and an alternative resolution procedure for them. As we said, a literal $\overline{L} \in \mathcal{Q}_0$ represents $-L$. Hence:

Definition 7.7. For a clause C we define $\underline{C}$ as the clause obtained by replacing every $\pm\overline{L}$ by the literal $\mp L$.

This is extended to sets of clauses as usual. Observe that if $S \models_{\mathrm{p}} C$ then $\underline{S} \models_{\mathrm{p}} \underline{C}$.

Definition 7.8. As in Section 6, $\mathsf{U} = \{f(x_1, \ldots, x_n) \mid f \in \Sigma, \text{ and each } x_i \in \mathbf{X}_r\}$. The functions eps and comp of Section 6 are now modified to return clauses of type C1 and C4 respectively. For a set S of clauses, define $\mathsf{ov}(S)$ as the set of clauses of type C2 in S, and $\mathsf{gr}(S)$ as the set of clauses of type C3 in S. The function π is as before.

We need to define which kinds of instantiations are to be used to generate propositional implications.

Definition 7.9. For a clause C, define

$$\begin{aligned}
\mathsf{I}_1(C) &= C[\mathsf{U}[\mathsf{Ngrr} \cup \mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G}_1]]]] \cup C[\mathsf{Ngr}_1] \cup C[\mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]]] \\
\mathsf{I}_2(C) &= \{C[\mathbf{x}_{r+1}]\} \cup C[\mathsf{Ngrr}[\mathsf{G}_1]] \\
\mathsf{I}_3(C) &= \{C\} \\
\mathsf{I}_4(C) &= \pi(C) \cup C[\mathsf{Ngrr} \cup \mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G}_1]]]
\end{aligned}$$

The instantiations defined by I_i are necessary for clauses of type Ci. Observe that $C[U] \subseteq \mathsf{I}_1(C)$. For a set S of clauses, define $\mathsf{I}_i(S) = \bigcup_{C \in S} \mathsf{I}_i(C)$. For a set S of clauses of type C1-C4 define $\mathsf{I}(S) = \mathsf{I}_1(\underline{\mathsf{eps}(S)}) \cup \mathsf{I}_2(\underline{\mathsf{ov}(S)}) \cup \mathsf{I}_3(\underline{\mathsf{gr}(S)}) \cup \mathsf{I}_4(\underline{\mathsf{comp}(S)}) \cup cl(\mathcal{R})[\mathsf{Ngrr} \cup \mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G}_1]]]$. Note that instantiations of clauses in $\underline{cl(\mathcal{R})}$ are necessary for the replacement rules, as argued above.

Definition 7.10. For a set T of clauses define the following properties:

—C satisfies property P1$_T$ iff $C[\mathbf{x}_{r+1}] \in T$.
—C satisfies property P2$_T$ iff $\mathsf{I}(T) \models_{\mathrm{p}} \mathsf{I}_2(\underline{C})$.
—C satisfies property P3$_T$ iff $\mathsf{I}(T) \models_{\mathrm{p}} \mathsf{I}_3(\underline{C})$.
—C satisfies property P4$_T$ iff $\mathsf{I}(T) \models_{\mathrm{p}} \mathsf{I}_4(\underline{C})$.

For sets of clauses S and T, define $S \sqsubseteq T$ to mean that every $C \in S$ is of type Ci and satisfies property Pi_T for some $1 \leq i \leq 4$. This is extended to tableaux as usual. We first consider the effect of one step of the above resolution procedure without splitting. This will help us to reuse this result in the Horn case in Section 8 where we use another variant of splitting. Accordingly let ϕ_0 be the variant of ϕ which applies replacement rules and $\mathcal{Q}_0$-splitting, but no ϵ-splitting.

PROPOSITION 7.11. *Let S be a set of clauses of type C1-C4. If $S \Rrightarrow_{\prec_s,\phi_0,\mathcal{R}} S'$ then one of the following statements holds.*

—$S' \sqsubseteq S$

—*$S' = S \cup \{C\} \cup S''$, C is a renaming of $B_1[\mathbf{x}_{i_1}] \sqcup \ldots \sqcup B_k[\mathbf{x}_{i_k}] \sqcup D$, each B_i is an ϵ-block, $1 \leq i_1, \ldots, i_k \leq r$, $D \subseteq \pm\mathcal{Q}$, $\mathsf{I}(S) \models_{\mathrm{p}} \underline{C}$, and S'' is a set of clauses of type C3 and $\emptyset \models_{\mathrm{p}} \underline{S''}$. If $k \geq 2$ then D has no literals $-q$ with $q \in \mathcal{Q} \setminus \mathcal{Q}_0$.*

PROOF. The set S'' in the second statement will contain the clauses $\overline{L} \vee L$ added by $\mathcal{Q}_0$-splitting, while C will be the clause produced by binary resolution or factoring, possibly followed by applications of replacement rules and by replacement of ground literals L by $-\overline{L}$. Hence $S'' = \emptyset$ in all cases except when we need to perform $\mathcal{Q}_0$-splitting.

First we consider resolution steps where splitting literals are resolved upon. A positive splitting literal cannot be chosen to resolve upon in a clause unless the clause has no literals other than positive splitting literals. Hence this clause is $C_1 = q \vee q_1 \vee \ldots \vee q_m$ of type C1, The other clause must be $C_2 = C_2' \vee -q$ of type Ci for some $1 \leq i \leq 4$. Resolution produces clause $C = C_2' \vee q_1 \vee \ldots \vee q_m$ of type Ci, and no replacement or splitting rules apply. We have $\{C_1, C_2\} \models_{\mathrm{p}} C$ and $\{\underline{C_1}, \underline{C_2}\} \models_{\mathrm{p}} \underline{C}$. Hence $\mathsf{I}(S) \supseteq \{\underline{C_1}\} \cup \mathsf{I}_i(\underline{C_2}) \models_{\mathrm{p}} \mathsf{I}_i(\underline{C})$. If $i = 1$ then the second statement of the lemma holds because $\mathsf{I}_i(\underline{C})$ contains a renaming of $\underline{C}$. If $i > 1$ then the first statement holds.

Now we consider binary resolution steps where no splitting literals are resolved upon. This is possible only when no negative splitting literals are present in the premises. Then the resolvent has no negative splitting literals. $\mathcal{Q}_0$-splitting may create negative splitting literals, but none of them are from $\mathcal{Q} \setminus \mathcal{Q}_0$. Hence the last part of the second statement of the lemma is always true. In the following $D, D_1, \ldots$ denote subsets of $\mathcal{Q}_0$. When we write $C \vee D$, it is implicit that C has no splitting literals. We have the following cases:

(1) We do resolution between two clauses C_1 and C_2 from S, both of type C1, and the resolvent C is of type C1. Hence no splitting or replacement rules apply, $S' = S \cup \{C\}$, $\mathsf{I}(S) \supseteq \{\underline{C_1}[\mathbf{x}_{r+1}], \underline{C_2}[\mathbf{x}_{r+1}]\} \models_{\mathrm{p}} \underline{C}[\mathbf{x}_{r+1}]$. Hence the second statement holds.

(2) We do resolution between a clause $C_1[\mathbf{x}_{r+1}] = C_1'[\mathbf{x}_{r+1}] \vee D_1 \vee \pm P(\mathbf{x}_{r+1})$, of type C1, and a clause $C_2[\mathbf{x}_{r+1}] = \mp P(t[\mathbf{x}_{r+1}]) \vee C_2'[\mathbf{x}_{r+1}] \vee D_2$, of type C2, both from S upto renaming, and the resolvent is $C[\mathbf{x}_{r+1}] = C_1'[t[\mathbf{x}_{r+1}]] \vee C_2'[\mathbf{x}_{r+1}] \vee D_1 \vee D_2$. By ordering constraints $t[\mathbf{x}_{r+1}] \in \mathsf{Ngr}_1$ is non-trivial. All literals in $C_1'[t[\mathbf{x}_{r+1}]] \vee C_2'[\mathbf{x}_{r+1}]$ are of the form $\pm' Q(t'[\mathbf{x}_{r+1}])$ with $t'[\mathbf{x}_{r+1}] \in \mathsf{Ngr}_1$. Hence no splitting or replacement rules apply and $S' = S \cup \{C\}$. $\underline{C_1}[\mathsf{Ngr}_1] \cup \{\underline{C_2}[\mathbf{x}_{r+1}]\} \models_{\mathrm{p}} \underline{C}[\mathbf{x}_{r+1}]$. Hence $\mathsf{I}(S) \supseteq \mathsf{I}_1(\underline{C_1}) \cup \mathsf{I}_2(\underline{C_2}) \supseteq \underline{C_1}[\mathsf{Ngr}_1] \cup \underline{C_1}[\mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]]] \cup \{\underline{C_2}[\mathbf{x}_{r+1}]\} \cup \underline{C_2}[\mathsf{Ngrr}[\mathsf{G}_1]] \models_{\mathrm{p}} \{\underline{C}[\mathbf{x}_{r+1}]\} \cup \underline{C}[\mathsf{Ngrr}[\mathsf{G}_1]] = \mathsf{I}_2(\underline{C}[\mathbf{x}_{r+1}])$. If C_1' is non-empty or C_2' has some non-trivial literal then $C[\mathbf{x}_{r+1}]$ is of type C2, $S' \sqsubseteq S$ and the first statement holds. If C_1' is empty and C_2' has only trivial literals, then C is of type C1 and the second statement holds.

(3) We do resolution between a clause $C_1[\mathbf{x}_{r+1}] = C_1'[\mathbf{x}_{r+1}] \vee D_1 \vee \pm P(\mathbf{x}_{r+1})$ of type C1, and a clause $C_2 = \mp P(t) \vee C_2' \vee D_2$ of type C3, both from S upto renaming, and the resolvent is $C = C_1'[t] \vee C_2' \vee D_1 \vee D_2$. We know that $t \in \mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]]$. Hence no splitting or replacement rules apply. (The first item of Definition 7.6 does not apply because C is ground and the second item does not apply because $t \in \mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]]$. $\mathcal{Q}_0$-splitting does not apply because C is ground.) We have $S' = S \cup \{C\}$. $\{C_1[t], C_2\} \models_{\mathrm{p}} C$. Hence $\mathsf{I}(S) \supseteq \mathsf{I}_1(\underline{C_1}[\mathbf{x}_{r+1}]) \cup \mathsf{I}_3(\underline{C_2}) \supseteq \underline{C_1}[\mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]]] \cup \{\underline{C_2}\} \models_{\mathrm{p}} \mathsf{I}_3(\underline{C}) = \{\underline{C}\}$. If C_1' or C_2' is non-empty. then $C[\mathbf{x}_{r+1}]$ is of type C3, $S' \sqsubseteq S$ and the first statement holds. If C_1' and C_2' are empty then C is of type C1 and the second statement holds.

(4) We do resolution between a clause $C_1[\mathbf{x}_{r+1}] = C_1'[\mathbf{x}_{r+1}] \vee D_1 \vee \pm P(\mathbf{x}_{r+1})$ of type C1, and a clause $C_2[\mathbf{x}_1, \ldots, \mathbf{x}_r] = \mp P(f(x_1, \ldots, x_n)) \vee C_2'[\mathbf{x}_1, \ldots, \mathbf{x}_r] \vee D_2$ of type C4,

both from S upto renaming, and the resolvent is $C[\mathbf{x}_1,\ldots,\mathbf{x}_r] = C_1'[f(x_1,\ldots,x_n)] \vee C_2'[\mathbf{x}_1,\ldots,\mathbf{x}_r] \vee D_1 \vee D_2$. (By ordering constraints we have chosen a non trivial literal from C_2 for resolution). No splitting or replacement rules apply and $S' = S \cup \{C\}$. We have $\underline{C_1}[\mathsf{U}] \cup \{\underline{C_2}[\mathbf{x}_1,\ldots,\mathbf{x}_r]\} \supseteq \{\underline{C_1}[f(x_1,\ldots,x_n)], \underline{C_2}[\mathbf{x}_1,\ldots,\mathbf{x}_r]\} \models_{\mathrm{p}} \underline{C}[\mathbf{x}_1,\ldots,\mathbf{x}_r]$. Hence $\underline{C_1}[\mathsf{U}] \cup \pi(\underline{C_2}[\mathbf{x}_1,\ldots,\mathbf{x}_r]) \models_{\mathrm{p}} \pi(\underline{C}[\mathbf{x}_1,\ldots,\mathbf{x}_r])$ and $\underline{C_1}[\mathsf{U}[\mathsf{Ngrr} \cup \mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G_1}]]] \cup \underline{C_2}[\mathsf{Ngrr} \cup \mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G_1}]]]) \models_{\mathrm{p}} \underline{C}[\mathsf{Ngrr} \cup \mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G_1}]]]$. Hence $\mathsf{I}(S) \supseteq \mathsf{I}_1(\underline{C_1}) \cup \mathsf{I}_4(\underline{C_2}) \models_{\mathrm{p}} \mathsf{I}_4(\underline{C})$.

—Suppose C_1' is non-empty or C_2' has some non-trivial literal. Then C is of type C4. The only trivial literals in $C[\mathbf{x}_1,\ldots,\mathbf{x}_r]$ are those in $C_2'[\mathbf{x}_1,\ldots,\mathbf{x}_r]$ and hence they involve predicates from $\mathbb{P}$. Hence $C[\mathbf{x}_1,\ldots,\mathbf{x}_r]$ is of type C4 and the first statement holds.

—Suppose C_1' is empty and C_2' has only trivial literals. Then $C[\mathbf{x}_1,\ldots,\mathbf{x}_r] = B_1[\mathbf{x}_{i_1}] \sqcup \ldots \sqcup B_k[\mathbf{x}_{i_k}] \vee D_1 \vee D_2$ where $1 \leq i_1,\ldots,i_k \leq r$, and each B_i is an ϵ-block. The second statement holds.

(5) We do resolution between a clause $C_1[\mathbf{x}_{r+1}] = C_1'[\mathbf{x}_{r+1}] \vee D_1 \vee \pm P(s[\mathbf{x}_{r+1}])$ and a clause $C_2[\mathbf{x}_{r+1}] = \mp P(t[\mathbf{x}_{r+1}]) \vee C_2'[\mathbf{x}_{r+1}] \vee D_2$, both of type C2, and both from S upto renaming, and the resolvent is $C[\mathbf{x}_{r+1}] = C_1'[\mathbf{x}_{r+1}]\sigma \vee C_2'[\mathbf{x}_{r+2}]\sigma \vee D_1 \vee D_2$ where $\sigma = mgu(s[\mathbf{x}_{r+1}], t[\mathbf{x}_{r+2}])$ (we renamed the second clause before resolution). We know that $s[\mathbf{x}_{r+1}], t[\mathbf{x}_{r+1}] \in \mathsf{Ngr_1}$, and by ordering constraints both s and t are non-trivial. By Lemma 5.1 one of the following cases holds:

—$\mathbf{x}_{r+1}\sigma = \mathbf{x}_{r+2}\sigma = \mathbf{x}_{r+1}$. $C[\mathbf{x}_{r+1}] = C_1'[\mathbf{x}_{r+1}] \vee C_2'[\mathbf{x}_{r+1}]$. Hence no splitting or replacement rules apply and $S' = S \cup \{C\}$. We have $\{C_1[\mathbf{x}_{r+1}], C_2[\mathbf{x}_{r+1}]\} \models_{\mathrm{p}} C[\mathbf{x}_{r+1}]$. Hence $\mathsf{I}_2(\underline{C_1}[\mathbf{x}_{r+1}]) \cup \mathsf{I}_2(\underline{C_2}[\mathbf{x}_{r+1}]) \models_{\mathrm{p}} \mathsf{I}_2(\underline{C}[\mathbf{x}_{r+1}]) \ni \underline{C}[\mathbf{x}_{r+1}]$. If C_1' or C_2' contains some non-trivial literal then $C[\mathbf{x}_{r+1}]$ is of type C2 and the first condition holds. If C_1' and C_2' contain only trivial literals then C is of type C1 and the second condition holds.

—$\mathbf{x}_{r+1}\sigma, \mathbf{x}_{r+2}\sigma \in \mathsf{Ngrr}[\mathsf{G}] \subseteq \mathsf{Ngrr}[\mathsf{G_1}]$. Then every literal in $C[\mathbf{x}_{r+1}]$ is of the form $\pm' Q(u)$ with $u \in \mathsf{Ngr_1}[\mathsf{Ngrr}[\mathsf{G_1}]]$. No splitting or replacement rules apply, as in case 3 above, and $S' = S \cup \{C\}$. $\mathsf{I}(S) \supseteq \underline{C_1}[\mathsf{Ngrr}[\mathsf{G_1}]] \cup \underline{C_2}[\mathsf{Ngrr}[\mathsf{G_1}]] \models_{\mathrm{p}} \{\underline{C}\} = \mathsf{I}_3(\underline{C})$. If C_1' or C_2' is non-empty then C is of type C3 and the first statement holds. If C_1' and C_2' are empty then C is of type C1 and the second statement holds.

(6) We do resolution between a clause $C_1[\mathbf{x}_{r+1}] = C_1'[\mathbf{x}_{r+1}] \vee D_1 \vee \pm P(s[\mathbf{x}_{r+1}])$ of type C2, and a ground clause $\mp P(t) \vee C_2' \vee D_2$ of type C3, both from S upto renaming, and the resolvent is $C = C_1'[\mathbf{x}_{r+1}]\sigma \vee C_2' \vee D_1 \vee D_2$ where σ is a unifier of $s[\mathbf{x}_{r+1}]$ and t. We know that $s[\mathbf{x}_{r+1}] \in \mathsf{Ngr_1}$, $t \in \mathsf{Ngr_1}[\mathsf{Ngrr}[\mathsf{G_1}]]$, and by ordering constraints, s is non-trivial. We have the following cases:

—$t \in \mathsf{G_1}$. Then $\mathbf{x}_{r+1}\sigma$ is a strict subterm of t hence $\mathbf{x}_{r+1}\sigma \in \mathsf{G} \subseteq \mathsf{Ngrr}[\mathsf{G_1}]$.

—$t \in \mathsf{Ngr_1}[\mathsf{Ngrr}[\mathsf{G_1}]] \setminus \mathsf{G_1}$. Hence we can assume that $t = t_1[t']$ for some non-trivial $t_1[\mathbf{x}_{r+1}] \in \mathsf{Ngr_1}$ and some $t' \in \mathsf{Ngrr}[\mathsf{G_1}]$. (Note that this can be assumed even when $t \in \mathsf{Ngrr}[\mathsf{G_1}] \setminus \mathsf{G_1}$ because in that case we can write $t = t_1[t''[t''']]$ where $t_1 \in \mathsf{Ngr} \subseteq \mathsf{Ngr_1}$, $t_1[t''[\mathbf{x}_{r+1}]] \in \mathsf{Ngrr}$ and $t''' \in \mathsf{G_1}$. Hence $t''[\mathbf{x}_{r+1}] \in \mathsf{Ngrr}$ and we can take $t' = t''[t''']$.) Let $s' = \mathbf{x}_{r+1}\sigma$. As $s[s'] = t_1[t']$ hence $s[\mathbf{x}_{r+1}]$ and $t_1[\mathbf{x}_{r+2}]$ have a unifier $\sigma = \{\mathbf{x}_{r+1} \mapsto s', \mathbf{x}_{r+2} \mapsto t'\}$. From Lemma 5.1, one of the following is true:

—$s[\mathbf{x}_{r+1}] = t_1[\mathbf{x}_{r+1}]$. Hence we have $\mathbf{x}_{r+1}\sigma = s' = t' \in \mathsf{Ngrr}[\mathsf{G_1}]$.

—$\mathbf{x}_{r+1}\sigma_1, \mathbf{x}_{r+2}\sigma_1$ is in $U[V]$ where

—U is the set of strict ground subterms of $\mathsf{Ngr_1}$, hence is contained in G, and

—V is the set of non-ground strict subterms of $\mathsf{Ngr_1}$, hence is contained in $\mathsf{Ngrr} \cup \mathsf{G}$.

Hence $\mathbf{x}_{r+1}\sigma_1, \mathbf{x}_{r+2}\sigma_1 \in \mathsf{Ngrr}[\mathsf{G}] \subseteq \mathsf{Ngrr}[\mathsf{G}_1]$. Hence $s' \in \mathsf{Ngrr}[\mathsf{G}_1]$.

In each case we have $\mathbf{x}_{r+1}\sigma = s' \in \mathsf{Ngrr}[\mathsf{G}_1]$. Hence all literals in $C_1'[\mathbf{x}_{r+1}]\sigma$ are of the form $\pm Q(t)$ with $t \in \mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]]$. All literals in C_2' are of the form $\pm' Q(t)$ with $t \in \mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]]$. Hence no splitting or replacement rules apply and $S' = S \cup \{C\}$. $\mathsf{I}(S) \supseteq \mathsf{I}_2(\underline{C_1}[\mathbf{x}_{r+1}]) \cup \mathsf{I}_3(\underline{C_2}) \supseteq \underline{C_1}[\mathsf{Ngrr}[\mathsf{G}_1]] \cup \{\underline{C_2}\} \vDash_{\mathrm{p}} \{\underline{C}\} = \mathsf{I}_3(\underline{C})$. If C_1' or C_2' is non-empty then C is of type C3 and the first statement holds. If C_1' and C_2' are empty then C is of type C1 and the second statement holds.

(7) We do resolution between a clause $C_1[\mathbf{x}_{r+1}] = C_1'[\mathbf{x}_{r+1}] \vee D_1 \vee \pm P(s[\mathbf{x}_{r+1}])$ of type C2, and a clause $C_2[\mathbf{x}_1, \ldots, \mathbf{x}_r] = \mp P(f(x_1, \ldots, x_n)) \vee C_2'[\mathbf{x}_1, \ldots, \mathbf{x}_r] \vee D_2$ of type C4, both from S upto renaming, and $\pm P(s[\mathbf{x}_{r+1}])$ and $\mp P(f(x_1, \ldots, x_n))$ are the literals resolved upon from the respective clauses. (By ordering constraints we have chosen a non-trivial literal to resolve upon in the second clause). By ordering constraints $s[\mathbf{x}_{r+1}] \in \mathsf{Ngr}_1$ is non-trivial. Hence we have the following two cases for $s[\mathbf{x}_{r+1}] = f(s_1[\mathbf{x}_{r+1}], \ldots, s_n[\mathbf{x}_{r+1}])$.

(a) We have some $1 \leq i, j \leq n$ such that $x_i = x_j$ but $s_i[\mathbf{x}_{r+1}] \neq s_j[\mathbf{x}_{r+1}]$. By Lemma 5.2, the only possible unifier of the terms $s[\mathbf{x}_{r+1}]$ and $f(x_1, \ldots, x_n)$ is σ such that $\mathbf{x}_{r+1}\sigma = g$ is a ground subterm of s_i or s_j and $x_k\sigma = s_k[g]$ for $1 \leq k \leq n$. As $s[\mathbf{x}_{r+1}] \in \mathsf{Ngr}_1$, we have $g \in \mathsf{G}$ and each $s_k[\mathbf{x}_{r+1}] \in \mathsf{Ngrr} \cup \mathsf{G}$. Hence $\mathbf{x}_{r+1}\sigma \in \mathsf{G}$ and each $x_k\sigma \in \mathsf{Ngrr}[\mathsf{G}] \cup \mathsf{G} \subseteq \mathsf{Ngrr}[\mathsf{G}_1]$. The resolvent $C = C_1'[\mathbf{x}_{r+1}]\sigma \cup C_2'[\mathbf{x}_1, \ldots, \mathbf{x}_r]\sigma \vee D_1 \vee D_2$ is ground. Each literal in $C_1'[\mathbf{x}_{r+1}]\sigma$ is of the form $\pm' Q(t)$ with $t \in \mathsf{Ngr}_1[\mathsf{G}] \subseteq \mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]]$. Each literal in $C_2'[\mathbf{x}_1, \ldots, \mathbf{x}_r]\sigma$ is of the form $\pm' Q(t)$ where the following cases can arise:

—$t = f'(x_{i_1}, \ldots, x_{i_m})\sigma$ such that $\{x_{i_1}, \ldots, x_{i_m}\} = \{x_1, \ldots, x_n\}$.
Then $t = f'(s_{i_1}, \ldots, s_{i_m})[g] \in \mathsf{Ngr}_1[\mathsf{G}_1] \subseteq \mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]]$.

—$t = x_k\sigma \in \mathsf{Ngrr}[\mathsf{G}_1] \subseteq \mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]]$ for some $1 \leq k \leq n$, where the literal $\pm' Q(x_k)$ is from C_2.

We conclude that all non-splitting literals in C are of the form $\pm' Q(t)$ with $t \in \mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]]$, and no splitting or replacement rules apply. We have $S' = S \cup \{C\}$. $\mathsf{I}(S) \supseteq \mathsf{I}_2(\underline{C_1}[\mathbf{x}_{r+1}]) \cup \mathsf{I}_4(\underline{C_2}[\mathbf{x}_1, \ldots, \mathbf{x}_r]) \supseteq \underline{C_1}[\mathsf{Ngrr}[\mathsf{G}_1]] \cup \underline{C_2}[\mathsf{Ngrr}[\mathsf{G}_1]] \vDash_{\mathrm{p}} \{C\} = \mathsf{I}_3(C)$. If C_1' or C_2' is non-empty then C is of type C3, and the first statement holds. If C_1' and C_2' are empty then C of type C1 and the second condition holds.

(b) For all $1 \leq i, j \leq n$, if $x_i = x_j$ then $s_i[\mathbf{x}_{r+1}] = s_j[\mathbf{x}_{r+1}]$.

Then $s[\mathbf{x}_{r+1}]$ and $f(x_1, \ldots, x_n)$ have mgu σ such that $x_k\sigma = s_k[\mathbf{x}_{r+1}] \in \mathsf{Ngrr} \cup \mathsf{G}$ for $1 \leq k \leq n$ and $x\sigma = x$ for $x \notin \{x_1, \ldots, x_n\}$. The resolvent $C[\mathbf{x}_{r+1}] = C_1'[\mathbf{x}_{r+1}] \vee C_2'\sigma \vee D_1 \vee D_2$ is a one-variable clause. $\{C_1[\mathbf{x}_{r+1}]\} \cup C_2[\mathsf{Ngrr} \cup \mathsf{G}] \vDash_{\mathrm{p}} C[\mathbf{x}_{r+1}]$. All literals in $C_1'[\mathbf{x}_{r+1}]$ are of the form $\pm' Q(t)$ with $t \in \mathsf{Ngr}_1$, and no replacement rules apply on them. All literals in $C_2'[\mathbf{x}_1, \ldots, \mathbf{x}_r]\sigma$ are of the form $\pm' Q(t[\mathbf{x}_{r+1}])$ where the following cases can arise:

—$t[\mathbf{x}_{r+1}] = f'(x_{i_1}, \ldots, x_{i_m})\sigma$ such that $\{x_{i_1}, \ldots, x_{i_m}\} = \{x_1, \ldots, x_n\}$. Then $t[\mathbf{x}_{r+1}] \in \mathsf{Ngr}_1$. No replacement rules apply on such a literal.

—$t[\mathbf{x}_{r+1}] = x_k\sigma = s_k[\mathbf{x}_{r+1}] \in \mathsf{Ngrr}$ for some $1 \leq k \leq n$, where the literal $\pm' Q(x_k)$ is from C_2. Hence we must have $Q \in \mathbb{P}$. Let $s_k[\mathbf{x}_{r+1}] = t_1[\ldots[t_p[\mathbf{x}_{r+1}]]\ldots]$ for some $p \geq 0$ where each $t_i[\mathbf{x}_{r+1}] \in \mathsf{Ngr}$ is non-trivial and irreducible. Such a literal is replaced by the literal $\pm' Q t_1 \ldots t_{p-1}(t_p[\mathbf{x}_{r+1}])$ and we know that $t_p \in \mathsf{Ngr} \subseteq \mathsf{Ngr}_1$. This new clause is obtained by propositional resolution between the former clause and clauses from $cl(\mathcal{R})[\mathsf{Ngrr}]$.

—$t[\mathbf{x}_{r+1}] = x_k\sigma = s_k \in \mathsf{G}$ for some $1 \leq k \leq n$, where the literal $\pm' Q(x_k)$ is from C_2. Hence we must have $Q \in \mathbb{P}$. No replacement rules apply on such a literal. If C contains only ground literals then this literal is left unchanged. Otherwise we perform $\mathcal{Q}_0$-splitting and this literal is replaced by the literal $-\pm' Q(s_k)$ and also a new clause $C'' = \pm' Q(s_k) \vee \pm' Q(s_k)$ of type C3 is added to S. If C' is the new clause obtained by this splitting then $\underline{C'}$ is clearly propositionally equivalent to the former clause. Also $\underline{C''} = \mp' Q(s_k) \vee \pm' Q(s_k)$ is a propositionally valid statement.

We conclude that after zero or more replacement and splitting rules, we obtain a clause $C'[\mathbf{x}_{r+1}]$, together with a set S'' of clauses of type C3, $\{\underline{C}[\mathbf{x}_{r+1}]\} \cup cl(\mathcal{R})[\mathsf{Ngrr}] \models_{\mathrm{p}} \{\underline{C'}[\mathbf{x}_{r+1}]\}$, $\emptyset \models_{\mathrm{p}} \underline{S''}$, and $S' = S \cup \{C'\} \cup S''$. $\{\underline{C_1}[\mathbf{x}_{r+1}]\} \cup \underline{C_2}[\mathsf{Ngrr} \cup \mathsf{G}] \cup cl(\mathcal{R})[\mathsf{Ngrr}] \models_{\mathrm{p}} \underline{C'}[\mathbf{x}_{r+1}]$. Hence $\mathsf{I}(S) \supseteq \mathsf{I}_2(\underline{C_1}) \cup \mathsf{I}_4(\underline{C_2}) \supseteq \{\underline{C_1}[\mathbf{x}_{r+1}]\} \cup \underline{C_1}[\mathsf{Ngrr}[\mathsf{G}_1]] \cup \underline{C_2}[\mathsf{Ngrr} \cup \mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G}_1]]] \cup cl(\mathcal{R})[\mathsf{Ngrr}] \cup cl(\mathcal{R})[\mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G}_1]]] \models_{\mathrm{p}} \mathsf{I}_2(\underline{C'}) \cup \mathsf{I}_3(\underline{S''}) = \underline{C'}[\mathbf{x}_{r+1}] \cup \underline{C'}[\mathsf{Ngrr}[\mathsf{G}_1]] \cup \underline{S''}$. If C' is of type C2 or C3 then the first statement holds. Otherwise C' is of type C1 and the second statement holds.

(8) We do resolution between a clause $C_1 = C_1' \vee D_1 \vee \pm P(s)$ and a clause $C_2 = \mp P(s) \vee C_2' \vee D_2$, both ground clauses of type C3 from S, and the resolvent is $C = C_1' \vee C_2' \vee D_1 \vee D_2$. No replacement or splitting rules apply and we have $S' = S \cup \{C\}$. $\mathsf{I}(S) \supseteq \{\underline{C_1}, \underline{C_2}\} \models_{\mathrm{p}} \mathsf{I}_3(\underline{C}) = \{\underline{C}\}$. If C_1' or C_2' is non-empty then C is of type C3, and the first statement holds. If C_1' and C_2' are empty then C is of type C1 and the second statement holds.

(9) We do resolution between a ground clause $C_1 = C_1' \vee D_1 \vee \pm P(s)$ of type C3, and a clause $C_2[\mathbf{x}_1, \ldots, \mathbf{x}_r] = \mp P(f(x_1, \ldots, x_n)) \vee C_2'[\mathbf{x}_1, \ldots, \mathbf{x}_r] \vee D_2$ of type C4, both from S upto renaming, and $\pm P(s)$ and $\mp P(f(x_1, \ldots, x_n))$ are the literals resolved upon from the respective clauses. We know that $s \in \mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]]$. Hence we have the following two cases for s.

(a) $s \in \mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]] \setminus \mathsf{G}_1$. Hence s must be of the form $f(s_1, \ldots, s_n)[g]$ for some $f(s_1, \ldots, s_n) \in \mathsf{Ngr}_1$ and some $g \in \mathsf{Ngrr}[\mathsf{G}_1]$ (The symbol f is same as in the literal $\mp P(f(x_1, \ldots, x_n))$ otherwise this resolution step would not be possible). We have each $s_i \in \mathsf{Ngrr} \cup \mathsf{G}$. The mgu σ of s and $f(x_1, \ldots, x_n)$ is such that $x_i\sigma = s_i[g] \in \mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G}_1]]$. The resolvent $C = C_1' \vee C_2'[\mathbf{x}_1, \ldots, \mathbf{x}_r]\sigma \vee D_1 \vee D_2$ is a ground clause. All literals in C_1' are of the form $\pm' Q(t)$ with $t \in \mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]]$ hence no replacement rules apply on them. The literals in $C_2'[\mathbf{x}_1, \ldots, \mathbf{x}_r]\sigma$ are of the form $\pm' Q(t)$ where the following cases are possible:

—$t = f'(x_{i_1}, \ldots, x_{i_m})\sigma$ where $\{x_{i_1}, \ldots, x_{i_m}\} = \{x_1, \ldots, x_n\}$. Then $f'(s_{i_1}, \ldots, s_{i_m}) \in \mathsf{Ngr}_1$. Hence $t \in \mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]]$. No replacement rules apply on such a literal.

—$t = x_i\sigma \in \mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G}_1]]$ for some $1 \leq i \leq n$. If $t \in \mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]]$ then no replacement rules apply on this literal. Otherwise we have $t \in \mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G}_1]] \setminus \mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]]$. We have $t = t_1[\ldots[t_p[t']]\ldots]$ for some irreducible non-trivial non-ground terms t_1, $\ldots, t_p \in \mathsf{Ngr}$ with $p \geq 0$ such that $t_1[\ldots[t_p[y]]] \in \mathsf{Ngrr}$ and $t' \in \mathsf{Ngrr}[\mathsf{G}_1]$, and the replacement strategy replaces this literal by the literal $\pm' Q t_1 \ldots t_{p-1}(t_p[t'])$, and we know that $t_p \in \mathsf{Ngr} \subseteq \mathsf{Ngr}_1$ so that $t_p[t'] \in \mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]]$. This new clause can be obtained by propositional resolution between the former clause and clauses from $cl(\mathcal{R})[\mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G}_1]]]$

We conclude that after zero or more replacement rules, we obtain a ground clause C', all of whose non-splitting literals are of the form $\pm' Q(t)$ with $t \in \mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]]$, and which is obtained by propositional resolution from $\{C\} \cup cl(\mathcal{R})[\mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G}_1]]]$. No splitting

rules apply and $S' = S \cup \{C'\}$. $\{\underline{C_1}\} \cup \underline{C_2}[\mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G_1}]]] \models_{\mathrm{p}} \underline{C}$ hence $\mathsf{I}(S) \supseteq \mathsf{I}_3(\underline{C_1}) \cup \mathsf{I}_4(\underline{C_2}) \cup cl(\mathcal{R})[\mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G_1}]]] \models_{\mathrm{p}} \mathsf{I}_3(\underline{C'}) = \{\underline{C'}\}$. If C'_1 or C'_2 is non-empty then C is of type C3, and the first statement holds. If C'_1 and C'_2 are empty then C is of type C1 and the second statement holds.

(b) $s \in \mathsf{G_1}$. For the resolution step to be possible we must have $s = f(s_1, \ldots, s_n)$. Each $s_i \in \mathsf{G}$. The mgu σ of s and $f(x_1, \ldots, x_n)$ is such that each $x_i\sigma = s_i$. The resolvent $C = C'_1 \vee C'_2[\mathbf{x}_1, \ldots, \mathbf{x}_r]\sigma \vee D_1 \vee D_2$ is a ground clause. All literals in C'_1 are of the form $\pm' Q(t)$ with $t \in \mathsf{Ngr_1}[\mathsf{Ngrr}[\mathsf{G_1}]]$. The literals in $C'_2[\mathbf{x}_1, \ldots, \mathbf{x}_r]\sigma$ are of the form $\pm' Q(t)$ where the following cases are possible:

—$t = f'(x_{i_1}, \ldots, x_{i_m})\sigma$ where $\{x_{i_1}, \ldots, x_{i_m}\} = \{x_1, \ldots, x_n\}$.
Then $t = f'(s_{i_1}, \ldots, s_{i_m}) \in \mathsf{G_1} \subseteq \mathsf{Ngr_1}[\mathsf{Ngrr}[\mathsf{G_1}]]$.

—$t = x_i\sigma = s_i \in \mathsf{G} \subseteq \mathsf{Ngr_1}[\mathsf{Ngrr}[\mathsf{G_1}]]$ for some $1 \leq i \leq n$.

Hence all non-splitting literals in C are of the form $\pm' Q(t)$ with $t \in \mathsf{Ngr_1}[\mathsf{Ngrr}[\mathsf{G_1}]]$. No replacement rules or splitting rules apply and $S' = S \cup \{C\}$. $\{C_1\} \cup C_2[\mathsf{G}] \models_{\mathrm{p}} C$ hence $\mathsf{I}(S) \models_{\mathrm{p}} \mathsf{I}_3(\underline{C}) = \{\underline{C}\}$. If C'_1 or C'_2 is non-empty then C is of type C3 and the first statement holds. If C'_1 and C'_2 are empty then C is of type C1 and the second statement holds.

(10) We do resolution between two clauses $C_1[\mathbf{x}_1, \ldots, \mathbf{x}_r]$ and $C_2[\mathbf{x}_1, \ldots, \mathbf{x}_r]$, both of type C4, and both from S upto renaming. We rename the second clause as $C_2[\mathbf{x}_{r+1}, \ldots, \mathbf{x}_{2r}]$ by applying the renaming $\sigma_0 = \{\mathbf{x}_1 \mapsto \mathbf{x}_{r+1}, \ldots, \mathbf{x}_r \mapsto \mathbf{x}_{2r}\}$. By ordering constraints, $C_1[\mathbf{x}_1, \ldots, \mathbf{x}_r] = C'_1[\mathbf{x}_1, \ldots, \mathbf{x}_r] \vee D_1 \vee P(f(x_1, \ldots, x_n))$ and $C_2[\mathbf{x}_{r+1}, \ldots, \mathbf{x}_{2r}] = -P(f(y_1, \ldots, y_n)) \vee C'_2[\mathbf{x}_{r+1}, \ldots, \mathbf{x}_{2r}] \vee D_2$ and the resolvent is $C[\mathbf{x}_1, \ldots, \mathbf{x}_r] = C'_1[\mathbf{x}_1, \ldots, \mathbf{x}_r]\sigma \vee C'_2[\mathbf{x}_{r+1}, \ldots, \mathbf{x}_{2r}]\sigma \vee D_1 \vee D_2$ where, by Lemma 6.2, σ is such that $\{x_1, \ldots, x_n\}\sigma \subseteq \{x_1, \ldots, x_n\}$ and $y_i\sigma = x_i\sigma$ for $1 \leq i \leq n$. $\pi(C_1) \cup \pi(C_2) \models_{\mathrm{p}} C[\mathbf{x}_1, \ldots, \mathbf{x}_r]$. Hence $\mathsf{I}(S) \supseteq \mathsf{I}_4(\underline{C_1}[\mathbf{x}_1, \ldots, \mathbf{x}_r]) \cup \mathsf{I}_4(\underline{C_2}[\mathbf{x}_1, \ldots, \mathbf{x}_r]) = \pi(\underline{C_1}[\mathbf{x}_1, \ldots, \mathbf{x}_r]) \cup \underline{C_1}[\mathsf{Ngrr} \cup \mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G_1}]]] \cup \pi(\underline{C_2}[\mathbf{x}_1, \ldots, \mathbf{x}_r]) \cup \underline{C_2}[\mathsf{Ngrr} \cup \mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G_1}]]] \models_{\mathrm{p}} \pi(\underline{C}[\mathbf{x}_1, \ldots, \mathbf{x}_r]) \cup \underline{C}[\mathsf{Ngrr} \cup \mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G_1}]]] = \mathsf{I}_4(\underline{C}[\mathbf{x}_1, \ldots, \mathbf{x}_r])$.

—Suppose C'_1 or C'_2 has a non-trivial literal. Then C is of type C4, no replacement or splitting rules apply, $S' = S \cup \{C\}$ and the first statement holds.

—Suppose C'_1 and C'_2 contain no non-trivial literal. Then $C[\mathbf{x}_1, \ldots, \mathbf{x}_r] = B_1[\mathbf{x}_{i_1}] \sqcup \ldots \sqcup B_k[\mathbf{x}_{i_k}] \vee D_1 \vee D_2$ with $1 \leq i_1, \ldots, i_k \leq r$, each B_i being an ϵ-block. No splitting or replacement rules apply (ϵ-splitting is forbidden by ϕ_0), and $S' = S \cup \{C\}$. The second statement holds.

(11) We do a resolution step in which one of the premises is a clause from $cl(\mathcal{R})$. Every clause in $cl(\mathcal{R})$ is of type C2. Also trivially $\mathsf{I}_2(C) \subseteq \mathsf{I}(T)$. Hence this case can be dealt with in the same way as in the case where one of the premises of resolution is a clause of type C2.

Next we consider factoring steps. Factoring on a clause of type C1 or C3 is possible only if the two involved literals are the same, hence this is equivalent to doing nothing.

(1) We do factoring on a clause $C_1[\mathbf{x}_{r+1}] = C'_1[\mathbf{x}_{r+1}] \vee \pm P(s[\mathbf{x}_{r+1}]) \vee \pm P(t[\mathbf{x}_{r+1}])$ of type C2, and from S upto renaming. We know that $s[\mathbf{x}_{r+1}], t[\mathbf{x}_{r+1}] \in \mathsf{Ngr_1}$, and by ordering constraints s and t are non trivial. The clause obtained is $C[\mathbf{x}_{r+1}] = C'_1[\mathbf{x}_{r+1}]\sigma \vee \pm P(s[\mathbf{x}_{r+1}])\sigma$ where σ is a unifier of $s[\mathbf{x}_{r+1}]$ and $t[\mathbf{x}_{r+1}]$. If $s[\mathbf{x}_{r+1}] \neq t[\mathbf{x}_{r+1}]$ then by Lemma 5.2 $\mathbf{x}_{r+1}\sigma$ is a ground strict subterm of s or t, hence $\mathbf{x}_{r+1}\sigma \in \mathsf{G} \subseteq \mathsf{Ngrr}[\mathsf{G_1}]$. Each literal in C is of the form $\pm' Q(t')$ where $t' \in \mathsf{Ngr_1}[\mathsf{Ngrr}[\mathsf{G_1}]]$. Hence C is of type C3. No

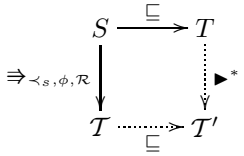
splitting or replacement rules apply and $S' = S \cup \{C\}$. We have $C \in C_1[\mathsf{Ngrr}[\mathsf{G}_1]]$. $\mathsf{I}(S) \supseteq \mathsf{I}_2(\underline{C_1}[\mathbf{x}_{r+1}]) \supseteq \underline{C_1}[\mathbf{x}_{r+1}][\mathsf{Ngrr}[\mathsf{G}_1]] \supseteq \mathsf{I}_3(C) = \{\underline{C}\}$. The first statement holds.

(2) We do factoring on a clause $C_1[\mathbf{x}_1, \ldots, \mathbf{x}_r]$ of type C4, and from S upto renaming, to obtain the clause $C[\mathbf{x}_1, \ldots, \mathbf{x}_r]$. By ordering constraints non-trivial literals must be chosen for factoring. Then $C[\mathbf{x}_1, \ldots, \mathbf{x}_r]$ is again of type C4 and $C[\mathbf{x}_1, \ldots, \mathbf{x}_r] \in \pi(C_1)$. $\mathsf{I}(S) \supseteq \mathsf{I}_4(\underline{C_1}) = \pi(\underline{C_1}) \cup \underline{C_1}[\mathsf{Ngrr} \cup \mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G}_1]]] \models_{\mathrm{p}} \mathsf{I}_4(\underline{C})$. The first statement holds.

□

The alternative resolution procedure for testing unsatisfiability by using succinct representations of tableaux is now defined by the rule: $\mathcal{T} \mid S \blacktriangleright \mathcal{T} \mid S \cup \{B_1 \sqcup D\} \mid S \cup \{B_2\} \mid \ldots \mid S \cup \{B_k\}$ whenever $\mathsf{I}(S) \models_{\mathrm{p}} B_1 \sqcup \ldots \sqcup B_k \sqcup \underline{D}$, each B_i is an ϵ-block, $1 \leq i_1, \ldots, i_k \leq r$ and $D \subseteq \pm\mathcal{Q}$. The simulation property now states:

LEMMA 7.12. *If $S \sqsubseteq T$ and $S \Rrightarrow_{\prec_s,\phi,\mathcal{R}} \mathcal{T}$ then $T \blacktriangleright^* \mathcal{T}'$ for some $\mathcal{T}'$ such that $\mathcal{T} \sqsubseteq \mathcal{T}'$.*



PROOF. As $S \Rrightarrow_{\prec_s,\phi,\mathcal{R}} \mathcal{T}$, we have some S' such that $S \Rrightarrow_{\prec_s,\phi_0,\mathcal{R}} S'$ and $\mathcal{T}$ is obtained from S' by ϵ-splitting steps. From Proposition 7.11, one of the following cases holds.

—$S' \sqsubseteq S$. Then S' contains only clauses of type C1-C4 and no ϵ-splitting is applicable. Hence $\mathcal{T} = S' \sqsubseteq S$. As $\mathcal{T} \sqsubseteq S$ and $S \sqsubseteq T$ hence $\mathcal{T} \sqsubseteq T$ because of transitivity of $\sqsubseteq$. Thus T is the required $\mathcal{T}'$.

—$S' = S \cup \{C\} \cup S''$, C is a renaming of $B_1[\mathbf{x}_{i_1}] \sqcup \ldots \sqcup B_k[\mathbf{x}_{i_k}] \sqcup D$ where each B_i is an ϵ-block, $1 \leq i_1, \ldots, i_k \leq r$, $D \subseteq \pm\mathcal{Q}$, $\mathsf{I}(S) \models_{\mathrm{p}} \underline{C}$ and S'' is a set of clauses of type C3 and $\emptyset \models_{\mathrm{p}} \underline{S''}$. We have $\mathcal{T} = S \cup S'' \cup \{B_1 \sqcup D\} \mid S \cup S'' \cup \{B_2\} \mid \ldots \mid S \cup S'' \cup \{B_k\}$. We have $S \cup S'' \cup \{B_1 \sqcup D\} \sqsubseteq T \cup \{B_1 \sqcup D\}$ and $S \cup S'' \cup \{B_i\} \sqsubseteq T \cup \{B_i\}$ for $1 \leq i \leq k$. We show that the required $\mathcal{T}'$ is $T \cup \{B_1 \sqcup D\} \mid T \cup \{B_1\} \mid \ldots \mid T \cup \{B_k\}$. As $S \sqsubseteq T$ hence $\mathsf{I}(T) \models_{\mathrm{p}} \mathsf{I}(S) \models_{\mathrm{p}} \underline{C}$. Hence $T \blacktriangleright \mathcal{T}'$.

□

Hence as for flat clauses we obtain:

THEOREM 7.13. *Satisfiability for the class $\mathcal{C}$ is NEXPTIME-complete.*

PROOF. Let S be a finite set in $\mathcal{C}$ whose satisfiability we want to show. We proceed as in the proof of Theorem 6.7. Wlog if $C \in S$ then C is either a complex clause or a one-variable clause. Clearly S is satisfiable iff $S \cup cl(\mathcal{R})$ is satisfiable. At the beginning we apply the replacement steps using $\mathcal{R}$ as long as possible and then $\mathcal{Q}_0$-splitting as long as possible. Hence wlog all clauses in S are of type C1-C4. Then we non-deterministically add a certain number of clauses of type C1 to S. Then we check that the resulting set S' does not contain $\Box$, and is saturated in the sense that: if $C = B_1[\mathbf{x}_{i_1}] \sqcup \ldots \sqcup B_k[\mathbf{x}_{i_k}] \sqcup D$, each B_i is an ϵ-block, $1 \leq i_1, \ldots, i_k \leq r$, $D \subseteq \pm\mathcal{Q}_0$, and $B_j[\mathbf{x}_{r+1}] \notin S'$ for some

$1 \leq j \leq k$, then $\mathsf{I}(S') \not\models_{\mathrm{p}} \underline{C}$. There are exponentially many such C to check for since the number of splitting literals is polynomial. The size of $\mathsf{I}(S')$ is exponential. □

8. THE HORN CASE

We show that in the Horn case, the upper bound can be improved to DEXPTIME. The essential idea is that propositional satisfiability of Horn clauses is in PTIME instead of NPTIME. But now we need to eliminate the use of tableaux altogether. To this end, we replace the ϵ-splitting rule of Section 7 by splitting-with-naming. Accordingly we instantiate the set $\mathcal{Q}$ used in Section 7 as $\mathcal{Q} = \mathcal{Q}_0 \cup \mathcal{Q}_1$ where $\mathcal{Q}_1 = \{\overline{C} \mid C$ is a non-empty negative ϵ-block with predicates from $\mathbb{P}\}$. We know that binary resolution and factorization on Horn clauses produces Horn clauses. Replacements on Horn clauses using the rules from $\mathcal{R}$ produces Horn clauses. $\mathcal{Q}_1$-splitting on Horn clauses produces Horn clauses. E.g. clause $P(\mathbf{x}_1) \vee -Q(\mathbf{x}_1) \vee -R(\mathbf{x}_2)$ produces $P(\mathbf{x}_1) \vee -Q(\mathbf{x}_1) \vee --\overline{R(\mathbf{x}_2)}$ and $\overline{-R(\mathbf{x}_2)} \vee -R(\mathbf{x}_2)$. $\mathcal{Q}_0$-splitting on $P(f(x)) \vee -Q(a)$ produces $P(f(\mathbf{x}_1)) \vee --\overline{Q(a)}$ and $\overline{-Q(a)} \vee -Q(a)$ which are Horn. However $\mathcal{Q}_0$-splitting on $C = -P(f(\mathbf{x}_1)) \vee Q(a)$ produces $C_1 = -P(f(\mathbf{x}_1)) \vee -\overline{Q(a)}$ and $C_2 = \overline{Q(a)} \vee Q(a)$. C_2 is not Horn. But $\underline{C_1} = C$ and $\underline{C_2} = -Q(a) \vee Q(a)$ are Horn. Finally, as $\mathcal{Q}_1$ has exponentially many atoms, we must restrict their occurrences in clauses. Accordingly, for $1 \leq i \leq 4$, define clauses of type Ci' to be clauses C of the type Ci, such that $\underline{C}$ is Horn and has at most r negative literals from $\mathcal{Q}_1$. ($\underline{C}$ is defined as before, hence it leaves atoms from $\mathcal{Q}_1$ unchanged). Now the $\mathcal{Q}$-splitting-replacement strategy ϕ_h first applies the replacement steps of Section 7 as long as possible, then applies $\mathcal{Q}_0$-splitting as long as possible and then applies $\mathcal{Q}_1$-splitting as long as possible. Succinct representations are now defined as: $S \sqsubseteq_h T$ iff for each $C \in S$, C is of type Ci' and satisfies Pi_T for some $1 \leq i \leq 4$. The abstract resolution procedure is defined as: $T \blacktriangleright_h T \cup \{B_1 \vee -q_2 \vee \ldots \vee -q_k \sqcup D \sqcup E\} \cup \{\overline{B_i} \vee B_i \mid 2 \leq i \leq k\}$ whenever $\mathsf{I}(T) \models_{\mathrm{p}} \underline{C}$, $C = B_1[\mathbf{x}_{i_1}] \sqcup \ldots \sqcup B_k[\mathbf{x}_{i_k}] \sqcup D \sqcup E$, $\underline{C}$ is Horn, $1 \leq i_1, \ldots, i_k \leq r$, B_1 is an ϵ-block, B_i is a negative ϵ-block and $2 \leq i \leq k$, $D \subseteq \pm\mathcal{Q}_0$ and $E \subseteq \pm\mathcal{Q}_1$ such that if $k = 1$ then E has at most r negative literals, and if $k > 1$ then E has no negative literal. The $\sqsubseteq$ and $\blacktriangleright$ relations are as in Section 7.

LEMMA 8.1. *If $S \sqsubseteq_h T$ and $S \Rrightarrow_{\prec_s, \phi_h, \mathcal{R}} S_1$ then $T \blacktriangleright_h{}^* T_1$ and $S_1 \sqsubseteq_h T_1$ for some T_1.*

PROOF. Let ϕ_0 be as in Section 7. As $S \Rrightarrow_{\prec_s, \phi_h, \mathcal{R}} S_1$ hence we have some S' such that $S \Rrightarrow_{\prec_s, \phi_0, \mathcal{R}} S'$ and S_1 is obtained from S' by applying $\mathcal{Q}_1$-splitting steps. As discussed above, all clauses $C \in S_1 \cup S'$ are such that $\underline{C}$ is also Horn. If S' is obtained by resolving upon splitting literals, then one of the premises must be just a positive splitting literal. The other premise has at most r literals of the form $-q$ with $q \in \mathcal{Q}_1$, hence the resolvent has at most r literals of the form $-q$ with $q \in \mathcal{Q}_1$. In case non-splitting literals are resolved upon then the premises cannot have any negative splitting literal and the resolvent has no negative splitting literal. $\mathcal{Q}_0$-splitting does not create literals from $\pm\mathcal{Q}_1$. Hence all clauses in S' have at most r literals of the form $-q$ with $q \in \mathcal{Q}_1$. Now by Proposition 7.11, one of the following conditions holds.

—$S' \sqsubseteq S$. Then $\mathcal{Q}_1$-splitting is not applicable on clauses in S' and $S_1 = S' \sqsubseteq S$. From transitivity of $\sqsubseteq$ we have $S_1 \sqsubseteq T$. Then from the above discussion we conclude that $S_1 \sqsubseteq_h T$.

—$S' = S \cup \{C\} \cup S''$, C is a renaming of $B_1[\mathbf{x}_{i_1}] \sqcup \ldots \sqcup B_k[\mathbf{x}_{i_k}] \sqcup D$, each B_i is an ϵ-block,

$1 \leq i_1, \ldots, i_k \leq r$, $D \subseteq \pm\mathcal{Q}$, $\mathsf{I}(S) \models_{\mathrm{p}} \underline{C}$, and S'' is a set of clauses of type C3 and $\emptyset \models_{\mathrm{p}} \underline{S''}$. Also if $k \geq 2$ then D has no literals $-q$ with $q \in \mathcal{Q}_1$. As C is Horn, wlog B_i is negative for $i \geq 2$. Hence $S_1 = S' \cup \{B_1 \vee -q_2 \vee \ldots \vee -q_k \sqcup D\} \cup \{\overline{B_i} \vee B_i \mid 2 \leq i \leq k\}$. We show that the required T_1 is $T \cup \{B_1 \vee -q_2 \vee \ldots \vee -q_k \sqcup D\} \cup \{\overline{B_i} \vee B_i \mid 2 \leq i \leq k\}$. Each $\overline{B_i} \cup B_i$ is of type C1'. As $C \in S'$ hence D has at most r literals $-q$ with $q \in \mathcal{Q}_1$. Hence if $k = 1$ then $B_1 \vee -q_2 \vee \ldots \vee -q_k \sqcup D$ is also of type C1'. If $k \geq 2$ then D has no negative literals $-q$ with $q \in \mathcal{Q}_1$, and $B_1 \vee -q_2 \vee \ldots \vee -q_k \sqcup D$ is again of type C1' since $k \leq r$. As $S \sqsubseteq_h T$ we have $\mathsf{I}(T) \models_{\mathrm{p}} \mathsf{I}(S) \models_{\mathrm{p}} \underline{C}$. Hence $T \blacktriangleright_h T_1$. Finally, clearly $S_1 \sqsubseteq T_1$ hence $S_1 \sqsubseteq_h T_1$.

□

Now for deciding satisfiability of a set of flat and one-variable clauses we proceed as in the non-Horn case. But now instead of non-deterministically adding clauses, we compute a sequence $S = S_0 \blacktriangleright_h S_1 \blacktriangleright_h S_2 \ldots$ starting from the given set S, and proceeding don't care non-deterministically, till no more clauses can be added, and then check whether $\Box$ has been generated. The length of this sequence is at most exponential. Computing S_{i+1} from S_i requires at most exponential time because the number of possibilities for C in the definition of $\blacktriangleright$ above is exponential in the size of S. (Note that this idea of $\mathcal{Q}_1$-splitting would not have helped in the non-Horn case because we cannot bound the number of positive splitting literals in a clause in the non-Horn case, whereas Horn clauses by definition have at most one positive literal). Also note that APDS can be encoded using flat Horn clauses. Hence:

THEOREM 8.2. *Satisfiability for the classes $\mathcal{CH}orn$ and $\mathcal{FH}orn$ is DEXPTIME-complete.*

Together with Theorem 4.1, this gives us optimal complexity for protocol verification:

THEOREM 8.3. *Secrecy of cryptographic protocols with single blind copying, with bounded number of nonces but unbounded number of sessions is DEXPTIME-complete.*

8.1 Alternative Normalization Procedure

While Theorem 8.2 gives us the optimum complexity for the Horn case, we outline here an alternative normalization procedure for deciding satisfiability in the Horn case, in the style of [Nielson et al. 2002]. Our goal is to show that the Horn case can be dealt with using simpler techniques. This may also be interesting for implementations, since it avoids exhaustive generation of instantiations of clauses. Define *normal* clauses to be clauses which have no function symbol in the body, have no repetition of variables in the body, and have no variables in the body other than those in the head. Sets of normal definite clauses involving unary predicates can be thought of as generalizations of tree automata, by adopting the convention that term t is *accepted* at *state* P iff atom $P(t)$ is reachable. I.e. states are just unary predicates. *(Intersection-)emptiness* and *membership* properties are defined as usual.

LEMMA 8.4. *Emptiness and membership properties are decidable in polynomial time for sets of normal definite clauses.*

PROOF. Let S be the set of clauses. To test emptiness of a state P, we remove arguments of predicate symbols in clauses, and treat predicates as proposition symbols. Then we add the clause $-P$ and check satisfiability of the resulting propositional Horn clause set.

To test if t is accepted at P, let T be the set of subterms of t. Define a set S' of clauses as follows. If $Q(s) \vee -Q_1(x_1) \vee \ldots \vee -Q_n(x_n) \in S$ and $s\sigma \in T$ for some substitution σ then we add the Horn clause $Q(s\sigma) \vee -Q_1(x_1\sigma) \vee \ldots \vee -Q_n(x_n\sigma)$ to S'. Finally we add $-P(t)$ to S' and test its unsatisfiability. S' is computable in polynomial time. Also S' has only ground clauses, hence satisfiability is equivalent to propositional unsatisfiability, by treating each ground literal as a propositional symbol.

□

For the rest of this section fix a set $\mathbb{S}$ of one-variable $\mathbb{P}$-clauses and flat $\mathbb{P}$-clauses. Let $\mathbb{Q}$, Ng, Ngs, Ngr, Ngrr, Ngr_1, G, and G_1 be defined as before with respect to this set $\mathbb{S}$ of clauses. We will be dealing only with $2^{\mathbb{Q}}$-clauses. The intuition behind the normalization procedure is as follows. We use states which are sets $\{P_1, P_2, \ldots, \} \subseteq \mathbb{Q}$, which intuitively represent the intersection of the states $P_1, P_2, \ldots$. These new states are denoted by $p, q, p_1, \ldots$. The state $\{P\}$ for $P \in \mathbb{Q}$ is also written as P. We try to make non-normal clauses redundant by resolving them with normal clauses. For example the normal clause $p_1(f(x)) \Leftarrow p_2(g(x))$ is resolved with the normal clause $p_2(g(x)) \Leftarrow p_3(x)$ to produce the clause $p_1(f(x)) \Leftarrow p_3(x)$. We also generate new clauses corresponding to conjunctions of states. For example $p_2(g(x)) \Leftarrow p_3(x)$ and $q_2(g(x)) \Leftarrow q_3(x)$ can be used to produce the clause $(p_2 \cup q_2)(g(x)) \Leftarrow p_3(x) \wedge q_3(x)$ which produces the clause $(p_2 \cup q_2)(g(x)) \Leftarrow (p_3 \cup q_3)(x)$.

For terms, literals and clauses M, the measure $||M||$ of M is defined as the number of distinct subterms of terms occurring in M. Let M be the maximum measure of the clauses occurring initially in $\mathbb{S}$. Let $\mathsf{M_f}$ be the maximum number of non-trivial literals in complex clauses occurring in $\mathbb{S}$. Let $\mathsf{M_o}$ be the maximum measure of the terms in Ngr_1. Wlog we assume that $\mathsf{M} \geq \mathsf{M_f} + \mathsf{M_o}$. If it were not the case, then we add a suitable one-variable clause to $\mathbb{S}$ to make it so. Also wlog $\mathsf{M_f} \geq r + 2$ and $r \geq 1$. We consider the following types of clauses

(C1") ϵ-clauses with predicates from $2^{\mathbb{Q}}$ and having at most M variables.

(C2") non-ground clauses which are renamings of one variable clauses with literals from $\pm 2^{\mathbb{Q}}(\mathsf{Ngr}_1) \cup \pm 2^{\mathbb{Q}}(\mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]])$. The measure of the clause is at most M.

(C3") ground clauses with literals from $\pm 2^{\mathbb{Q}}(\mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]])$, and having at most $\mathsf{M} + r$ literals.

(C4") complex clauses $\bigvee_{i=1}^{k} \pm_i p_i(f_i(x_1^i, \ldots, x_{n_i}^i)) \vee \bigvee_{j=1}^{l} \pm_j q_j(x_j)$, with each $p_i \subseteq \mathbb{Q}$, $k \leq \mathsf{M_f}$, each $n_i \geq 2$, each $q_j \subseteq \mathbb{P}$.

We consider $pt_1 \ldots t_n$ to be an abbreviation of the predicate $\{Pt_1, \ldots t_n \mid P \in p\} \subseteq \mathbb{Q}$ where $p \subseteq \mathbb{P}$. Let $\mathbf{Cl}$ be the set of clauses $pt_1 \ldots t_{i-1}(t_i[x]) \Leftarrow pt_1 \ldots t_i(x)$ and $pt_1 \ldots t_i(x) \Leftarrow pt_1 \ldots t_{i-1}(t_i[x])$ where $p \subseteq \mathbb{P}$ and $t_1 \ldots t_i \in \mathsf{Ngrr}$. Let $\mathbb{I}_1$ be the set or clauses of the form $(p_1 \cup p_2)(x) \Leftarrow p_1(x) \wedge p_2(x)$ where $\emptyset \neq p_1, p_2 \subseteq \mathbb{Q}$ and $p_1 \neq p_2$. Let $\mathbb{I}_2$ be the set of clauses of the form $p_1(x) \Leftarrow (p_1 \cup p_2)(x)$ for the same p_1 and p_2, and $\mathbb{I} = \mathbb{I}_1 \cup \mathbb{I}_2$.

We use the following normalization steps. The fourth step is called *eager*. During normalization, we always apply eager steps on the current clause set, whenever any are applicable, before proceeding.

(1) If clause $C \vee -p_1(t) \vee -p_2(t)$ is present in the current clause set then we add the clause $C \vee -(p_1 \cup p_2)(t)$.

(2) If clause $C \vee -p(x)$ is present in the current clause set then we add the clause C, provided that p is non-empty from the current set of normal clauses and that x does not occur in C.

(3) If clause $C \vee -p(t)$ is present in the current clause set then we add the clause C, provided that t is ground and is accepted at p from the current set of normal clauses.

(4) Let $p \subseteq \mathbb{P}$, $t_1[\ldots[t_n[x]]\ldots] \in \mathsf{Ngrr}$, $g \in \mathsf{Ngrr}[\mathsf{G_1}]$ and $t_{n-1}, t_n \in \mathsf{Ngr}$ are non-trivial. Clause $\pm p(t_1[\ldots[t_n[x]]\ldots]) \vee C$ of type C2"is eagerly replaced by the clause $\pm p t_1 \ldots t_{n-1}(t_n[x]) \vee C$. Clause $\pm p(t_1[\ldots[t_n[g]]\ldots]) \vee C$ is eagerly replaced by the clause $\pm p t_1 \ldots t_{n-1}(t_n[g]) \vee C$ if $t_1[\ldots[t_n[g]]\ldots] \in \mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G_1}]] \setminus \mathsf{Ngr_1}[\mathsf{Ngrr}[\mathsf{G_1}]]$.

(5) Suppose $p(s) \vee C$ is a normal clause and $q(x) \Leftarrow p(x)$ an ϵ-clause, both present in the current clause set. Then the clause $q(s) \vee C$ is added to the clause set.

(6) Suppose clause $C \vee -p(t)$ is present in the current clause set and t is a non-ground functional term, and $p(s) \vee D$ is a normal clause, other than an ϵ-clause, present in the current clause set, and both clauses are renamed so as not to share any variables. Also suppose that the first normalization step above is not applicable on $C \vee -p(t)$. Then the clause $(C \vee D)\sigma$ is added to the clause set where σ is mgu of s and t.

(7) Suppose $p_1(s) \vee C$ and $p_2(t) \vee D$ are two normal clauses other than ϵ-clauses, renamed so as not to share any variables. Then the clause $(p_1 \vee p_2)(s\sigma) \vee C\sigma \vee D\sigma$ is added where $\sigma = mgu(s, t)$.

LEMMA 8.5. *Let clause set S_2 is obtained from clause set S_1 by one normalization step, and $\mathbb{I} \cup \mathbf{Cl} \subseteq S_1$. Then $\mathbb{I} \cup \mathbf{Cl} \subseteq S_2$ and exactly the same set of ground atoms are derivable from S_1 and S_2.*

PROOF. No clause, other than those of type C2", ever get deleted, hence trivially $\mathbb{I} \cup \mathbf{Cl} \subseteq S_2$.

Except for the eager replacement step, the other steps involve adding a new clause. Because of the presence of clauses from $\mathbb{I}$ this new clause can always be obtained by resolution steps between suitable instances of old clauses (without any ordering constraints between the clauses, i.e. w.r.t. the empty ordering). Hence they do not let us derive anything new.

The eager replacement step involves replacing a clause C_1 by a clause C_2. C_1 can be obtained by resolution steps between clauses from $\mathbf{Cl} \cup \{C_2\}$. Similarly C_2 can be obtained by resolution steps between clauses from $\mathbf{Cl} \cup \{C_1\}$. □

LEMMA 8.6. *Let S be a set of clauses of the form (C1"-C4") with $\mathbb{I} \cup \mathbf{Cl} \subseteq S$. Let S_1 be the set of clauses produced by a normalization step followed by eager steps as long as they are applicable. Then S_1 also has only clauses of the form (C1"-C4").*

PROOF. The result is easy for the first three and the fifth normalization steps. The fourth step is not applicable. We now consider the sixth step. The unifications involved are as in the previous sections, the new arguments here being to take care of the size restrictions. Let C_1 be the non-normal clause and C_2 the normal clause involved. Note that because the first kind of normalization step is not applicable on C_1 hence C_1 does not have literals $-p_1(t)$ and $-p_2(t)$ with $p_1 \neq p_2$. Hence if C_1 is of type C2", then it has at most $\mathsf{M} + 1$ literals. We have the following cases. We use the notation $C\ [\vee \pm p(t)]$ to denote a clause which is either C or $C \vee p(t)$.

(1) $C_1[\mathbf{x}_{r+1}] = C_1'[\mathbf{x}_{r+1}] \vee -p(s[\mathbf{x}_{r+1}])$ and $C_2[\mathbf{x}_{r+1}] = p(t[\mathbf{x}_{r+1}])\ [\vee -q(\mathbf{x}_{r+1})]$ are both of type C2", with $s[\mathbf{x}_{r+1}], t[\mathbf{x}_{r+1}] \in \mathsf{Ngr_1}$, and s and t are non-trivial. If

$s[\mathbf{x}_{r+1}] = t[\mathbf{x}_{r+1}]$ then there is nothing to show. Otherwise as in case 5 of Proposition 7.11, we get a ground clause C all of whose literals are from the set $\pm 2^{\mathbb{Q}}(\mathsf{Ngr_1}[\mathsf{Ngrr}[\mathsf{G_1}]])$. C has fewer literals than C_1 which has at most $\mathsf{M}+1$ literals.

(2) $C_1[\mathbf{x}_{r+1}] = C_1'[\mathbf{x}_{r+1}] \vee -p(s[\mathbf{x}_{r+1}])$ is of type C2", $s[\mathbf{x}_{r+1}] \in \mathsf{Ngr_1}$ is non-trivial, and $C_2 = p(t)$ with $t \in \mathsf{Ngr_1}[\mathsf{Ngrr}[\mathsf{G_1}]]$. As in case 6 of Proposition 7.11, we get a ground clause C having only literals from the set $\pm 2^{\mathbb{Q}}(\mathsf{Ngr_1}[\mathsf{Ngrr}[\mathsf{G_1}]])$. The number of literals in C is less than in C_1.

(3) We have $C_1[\mathbf{x}_{r+1}] = C_1'[\mathbf{x}_{r+1}] \vee -p(s[\mathbf{x}_{r+1}])$ of type C2", with $s[\mathbf{x}_{r+1}] = f(s_1[\mathbf{x}_{r+1}], \ldots, s_n[\mathbf{x}_{r+1}]) \in \mathsf{Ngr_1}$, and $C_2[\mathbf{x}_1, \ldots, \mathbf{x}_r] = p(f(x_1, \ldots, x_n)) \Leftarrow p_1(x_{i_1}) \wedge \ldots \wedge p_k(x_{i_k})$ is of the form C4", such that the x_{i_j} are the pairwise distinct. We have the following cases.

(a) $s_i[\mathbf{x}_{r+1}] = s_j[\mathbf{x}_{r+1}]$ whenever $x_i = x_j$ with $1 \leq i, j \leq n$. Then we get the new clause $C[\mathbf{x}_{r+1}] = C_1'[\mathbf{x}_{r+1}] \vee -p_1(s_{i_1}[\mathbf{x}_{r+1}]) \vee \ldots \vee -p_k(s_{i_k}[\mathbf{x}_{r+1}])$. All terms occurring in C occur already in C_1. If some s_i is non-ground then replacement rules may apply as in case 7 of Proposition 7.11, which do not create any new subterms. The resulting clause is of the right form, which is shown as in case 7 of Proposition 7.11.

(b) There is some $x_i = x_j$ and $s_i[\mathbf{x}_{r+1}] \neq s_j[\mathbf{x}_{r+1}]$. As in case 7 of Proposition 7.11, the new clause C is ground and has only literals from $\pm 2^{\mathbb{Q}}(\mathsf{Ngr_1}[\mathsf{Ngrr}[\mathsf{G_1}]])$. C_1 has at most $\mathsf{M}+1$ literals hence C has at most $\mathsf{M}+r$ literals.

(4) $C_1[\mathbf{x}_{r+1}, \ldots, \mathbf{x}_r] = C_1' \vee -p(f(x_1, \ldots, x_n))$ is of form C4", and $C_2[\mathbf{x}_{r+1}] = p(s[\mathbf{x}_{r+1}])\ [\vee -q(\mathbf{x}_{r+1})]$ is of type C2", with $s[\mathbf{x}_{r+1}] = f(s_1[\mathbf{x}_{r+1}], \ldots, s_n[\mathbf{x}_{r+1}])$. We again have the following cases.

—$s_i[\mathbf{x}_{r+1}] = s_j[\mathbf{x}_{r+1}]$ whenever $x_i = x_j$ with $1 \leq i, j \leq n$. Then we get the new clause $C[\mathbf{x}_{r+1}] = C_1'[s_1[\mathbf{x}_{r+1}], \ldots, s_n[\mathbf{x}_{r+1}]]\ [\vee -q(\mathbf{x}_{r+1})]$. We have $||C|| \leq \mathsf{M_f} + ||s|| \leq \mathsf{M_f} + \mathsf{M_o} \leq \mathsf{M}$. As in case 7 of Proposition 7.11 the new clause may require further replacements which do not create any new subterms.

—We have some $x_i = x_j$ but $s_i[\mathbf{x}_{r+1}] \neq s_j[\mathbf{x}_{r+1}]$. We get a ground clause all of whose literals are from $\pm 2^{\mathbb{Q}}(\mathsf{Ngr_1}[\mathsf{Ngrr}[\mathsf{G_1}]])$. The number of literals is less than in C_1. Also C_1 has at most $\mathsf{M}+r$ literals since at most $r-1$ variables occur in it, at most $\mathsf{M_f}$ non-trivial literals occur in it, and the first normalization step is not applicable on it.

(5) $C_1[\mathbf{x}_1, \ldots, \mathbf{x}_r] = C_1' \vee -p(f(x_1, \ldots, x_n))$ is of form C4", and $C_2 = p(s)$ where $s \in \mathsf{Ngr_1}[\mathsf{Ngrr}[\mathsf{G_1}]]$. We get a ground clause with fewer literals than C_1, and all of whose arguments are from $\pm 2^{\mathbb{Q}}(\mathsf{Ngr_1}[\mathsf{Ngrr}[\mathsf{G_1}]]) \cup \pm 2^{\mathbb{Q}}(\mathsf{Ngrr}[\mathsf{Ngrr}[\mathsf{G_1}]])$. Also C_1 has at most $\mathsf{M}+r$ literals. Some eager replacement steps may apply as in case 9 of Proposition 7.11.

(6) C_1 is the clause $C_1' \vee -p(f(x_1, \ldots, x_n))$ of form C4", and C_2 is the clause $p(f(y_1, \ldots, y_n)) \Leftarrow p_1(y_{i_1}) \wedge \ldots \wedge p_k(y_{i_k})$ is of the form C4". The normalization step produces a flat clause clause with at most r variables, and has fewer non-trivial literals that C_1.

Next we consider normalization steps of the seventh type.

(1) The two normal clauses involved are $C_1[\mathbf{x}_{r+1}] = p_1(s[\mathbf{x}_{r+1}])\ [\vee -q_1(\mathbf{x}_{r+1})]$ and $C_2[\mathbf{x}_{r+1}] = p_2(t[\mathbf{x}_{r+1}])\ [\vee -q_2(\mathbf{x}_{r+1})]$ of type C2", with s and t being non-trivial. If $s[\mathbf{x}_{r+1}] = t[\mathbf{x}_{r+1}]$ then we get a clause C with $||C|| \leq \mathsf{M_o} \leq \mathsf{M}$. Otherwise we get a ground clause of the required form with at most three literals.

(2) We have a normal clause $C_1[\mathbf{x}_{r+1}] = p_1(s[\mathbf{x}_{r+1}])\ [\vee - q_1(\mathbf{x}_{r+1})]$ of type C2", and a ground clause $C_2 = p_2(t)$ with $t \in \mathsf{Ngr}_1[\mathsf{Ngrr}[\mathsf{G}_1]]$. We get a ground clause of the required form with at most two literals.

(3) We have a normal clause $C_1[\mathbf{x}_{r+1}] = p_1(s[\mathbf{x}_{r+1}])\ [\vee - q_1(\mathbf{x}_{r+1})]$ of the form C2", $s[\mathbf{x}_{r+1}] = f(s_1[\mathbf{x}_{r+1}], \ldots, s_n[\mathbf{x}_{r+1}])$, and a normal clause $C_2[\mathbf{x}_1, \ldots, \mathbf{x}_r] = p(f(x_1, \ldots, x_n)) \Leftarrow p_1(y_1) \wedge \ldots \wedge p_k(y_k)$ is of type C4". Again we have the following two cases.
—For all $1 \leq i, j \leq n$, if $x_i = x_j$ then $s_i[\mathbf{x}_{r+1}] = s_j[\mathbf{x}_{r+1}]$. After some eager replacement steps, we get a one variable clause which has no occurrences of terms which do not occur already in C_1.
—There is some $x_i = x_j$ such that $s_i[\mathbf{x}_{r+1}] \neq s_j[\mathbf{x}_{r+1}]$. We get a ground clause of the right form and having at most $r + 2 \leq \mathsf{M}$ literals.

(4) We have a normal ground clause $C_1 = p_1(t)$ and another normal clause $C_2 = p_2(f(x_1, \ldots, x_n)) \Leftarrow q_1(x_{i_1}) \wedge \ldots \wedge q_k(x_{i_k})$. After some eager replacement steps, we get a ground clause C of the right form with at most $r + 1$ literals.

(5) The two normal clauses are ground clauses $p(t)$ and $q(t)$. The new clause is $(p \cup q)(t)$.

(6) The two normal clauses are complex clauses. We get a new complex clause with at most one non-trivial literal.

□

We call a set saturated when application of any of the above normalization steps (always followed by the eager steps as long as applicable) leaves the set unchanged.

LEMMA 8.7. *Let S be a saturated set of clauses with $\mathbb{I} \cup \mathbf{Cl} \subseteq S$. Then every atom derivable from the set S is derivable from the normal clauses other than ϵ-clauses in S.*

PROOF. Define the measure of a derivation to be (n_1, n_2) where n_1 is the number of applications of clauses of type C4", and n_2 is the total number of applications of clauses. We consider the lexicographic ordering on this measure. We show that any derivation involving a non-normal clause or an ϵ-clause can be transformed into a derivation of the same atom but which has strictly smaller measure. It suffices to consider a derivation which uses a non-normal clause or an ϵ-clause only at the last step. We have the following cases.

(1) Suppose clause $C \vee -q(x)$ is used as the last step where x does not occur in C. But then C must be present in the clause set, and we get a derivation involving strictly fewer steps. Also C cannot be of type C4".

(2) Suppose a clause $C \vee -q(s)$ is used as the last step to derive some $p(t)$, where s is ground. But then $q(s)$ is derivable using the normal clauses. Hence the clause C must be present in the clause set, and we get a derivation involving strictly fewer steps. Also C cannot be of type C4".

(3) A clause $p(x) \Leftarrow q(x)$ is used as the last step to derive the atom $p(t)$. The second last step involves application of some normal clause $q(s) \vee C$. Since the clause set is saturated, the last two steps can be replaced by an application of the clause $p(s) \vee C$. $p(s) \vee C$ is of type C4"only if $q(s) \vee C$ is of type C4".

(4) Suppose a clause $C_1 \vee -q(s_1)$ is used as the last step where s is non-ground and functional. Let σ_1 be the ground substitution used at the last step. The derivation of $q(s\sigma_1)$ uses some normal clause $q(s_2) \vee C_2$ as the last step and some substitution σ_2

as the last step, so that $s_1\sigma_1 = s_2\sigma_2$. Assume that the two clauses have been renamed so as not to share any variables. Then clearly we have a normalization step of the sixth type to produce some clause $C = C_1\sigma \vee C_2\sigma$. By definition of most general unifiers, we also have a substitution σ_3 such that $C_1\sigma\sigma_3 = C_1\sigma_1$ and $C_2\sigma\sigma_3 = C_2\sigma_2$. Hence by applying this new clause we get a derivation with strictly fewer steps. Also this new clause is of type C4"only if both the original clauses are of type C4", in which case, the new derivation will involve strictly fewer applications of clauses of type C4". In case this new clause is a clause of type C2"produced from a clause of type C2"and a clause of type C4", then some eager replacement steps may apply. In that case, given the clauses of **Cl**, we still have a derivation with strictly fewer number of applications of clauses of type C4".

(5) Suppose the last step involves an application of a clause $C \vee -p_1(x) \vee -p_2(x)$ and a substitution σ, with $x\sigma = s$. Suppose the derivation of $p_i(s)$ uses a clause $p_i(s_i) \vee C_i$ as the last step with substitution σ_i, for $i = 1, 2$. Assume that these two clauses have been renamed apart. s_1 and s_2 are unifiable and we have a mgu σ_3. We get the new clause $C_3 = (p_1 \cup p_2)(s_1\sigma_3) \vee C_1\sigma \vee C_2\sigma_3$. Also we have a ground substitution σ_4 such that $s_1\sigma_3\sigma_4 = s_1\sigma_1 = s = s_2\sigma_2 = s_2\sigma_3\sigma_4$, $C_1\sigma_3\sigma_4 = C_1\sigma_1$ and $C_2\sigma_3\sigma_4 = C_2\sigma_2$. Hence we get a new derivation with strictly fewer clauses. Also the new clause C_3 is of type C4"only if both $p_1(s_1) \vee C_1$ and $p_2(s_2) \vee C_2$ are of type C4", in which case, the new derivation will involve strictly fewer applications of clauses of type C4".
Eager replacement steps may apply in case the original clauses were of type C2"and C4"respectively (or vice versa), so that C_2 is of type C4". In that case, thanks to clauses from **Cl**, we still get a new derivation involving strictly fewer clauses of type C4".

□

We can now show the required result.

THEOREM 8.8. *A set of flat and one-variable clauses can be normalized in exponential time.*

PROOF. We have the set $\mathbb{S}$ of input $\mathbb{P}$-clauses. We add to $\mathbb{S}$ the clauses from $\mathbb{I}$ and **Cl**. This does not affect the set of derivable atoms of the form $P(t)$ with $P \in \mathbb{P}$. We then perform eager replacements. Now all clauses are of the form C1"-C4".

We now apply the normalization steps till no new clauses can be added. If we obtain a saturated set then we can remove the non-normal clauses and ϵ-clauses from it to get the required set of normal clauses. We next show that we get a saturated set after generating only exponentially many clauses.

First we consider consider clauses without literals of the form $-p(t)$ and $-q(t)$ with $p \neq q$. Then because of the form of clauses, only linear number of literals are possible in any clause. Since we have exponentially many predicates and exponentially many terms, this gives us an exponential bound on the number of clauses.

Next we consider clauses which may have duplications of the above form. If the clause was in the clause set at the beginning, then it has only linear number of literals. Otherwise it was produced by a normalization step. This first five normalization steps never increase the number of literals. The sixth and seventh step only use two clauses without repetitions of the above form, and produce a clause having at most double the number of literals in

either of the two clauses. Hence the number of literals is again linear.

Hence we generate only exponentially many clauses of linear size. This also means that the number of emptiness and membership tests performed is only exponentially many. □

Example 1. Consider the set $S = \{C_1, \ldots, C_5\}$ of clauses where

$$\begin{array}{rl} C_1 = & P(a) \\ C_2 = & Q(a) \\ C_3 = & P(f(g(\mathbf{x}_1, a), g(a, \mathbf{x}_1), a)) \ \vee - P(\mathbf{x}_1) \\ C_4 = & P(f(g(\mathbf{x}_1, a), g(a, \mathbf{x}_1), b)) \ \vee - P(\mathbf{x}_1) \\ C_5 = & R(\mathbf{x}_1) \ \vee - P(f(\mathbf{x}_1, \mathbf{x}_1, \mathbf{x}_2)) \vee -Q(\mathbf{x}_2) \end{array}$$

C_5 is not normal. Resolving it with C_3 gives the clause

$$R(g(a, a)) \vee -P(a) \vee -Q(a)$$

As a is accepted at P and Q using the normal clauses C_1' and C_2', hence we get a new normal clause

$$C_6 = R(g(a, a))$$

Resolving C_5 with C_4 gives

$$R(g(a, a)) \vee -P(a) \vee -Q(b)$$

But b is not accepted at Q using the normal clauses hence this clause is rejected. Finally C_1 and C_2 also give the normal clause

$$C_7 = \{P, Q\}(a)$$

The resulting set of normal clauses is $\{C_1, \ldots, C_4, C_6, C_7\}$.

9. CONCLUSION

We have proved DEXPTIME-hardness of secrecy for cryptographic protocols with single blind copying, and have improved the upper bound from 3-DEXPTIME to DEXPTIME. We have improved the 3-DEXPTIME upper bound for satisfiability for the class $\mathcal{C}$ to NEXPTIME in the general case and DEXPTIME in the Horn case, which match known lower bounds. For this we have invented new resolution techniques like ordered resolution with splitting modulo propositional reasoning, ordered literal replacements and decompositions of one-variable terms. As byproducts we obtained optimum complexity for several fragments of $\mathcal{C}$ involving flat and one-variable clauses. For implementation purposes we have also given an exponential time normalization procedure to transform such clauses sets into normal form on which various queries can be efficiently answered. Security for several other decidable classes of protocols with unbounded number of sessions and bounded number of nonces is in DEXPTIME, suggesting that DEXPTIME is a reasonable complexity class for such classes of protocols.

APPENDIX

A. PROOFS OF SECTION 5

We use the following unification algorithm, due to Martelli and Montanari. It is described by the following rewrite rules on finite multisets of equations between terms; we let M be any such multiset, and comma denote multiset union:

(Delete). $M, u \doteq u \to M$

(Decomp). $M, f(u_1, \ldots, u_n) \doteq f(v_1, \ldots, v_n) \to M, u_1 \doteq v_1, \ldots, u_n \doteq v_n$

(Bind). $M, x \doteq v \to M[x := v], x \doteq v$ provided x is not free in v, but is free in M.

(Fail1). $M, x \doteq v \to \bot$ provided x is free in v and $x \neq v$.

(Fail2). $M, f(u_1, \ldots, u_m) \doteq g(v_1, \ldots, v_n) \to \bot$ provided $f \neq g$.

We consider that equations $u \doteq v$ are unordered pairs of terms u, v, so that in particular $u \doteq v$ and $v \doteq u$ are the same equation. $\bot$ represents failure of unification. If s and t are unifiable, then this rewrite process terminates, starting from $s \doteq t$, on a so-called solved form $z_1 \doteq u_1, \ldots, z_k \doteq u_k$; then $\sigma = \{z_1 \mapsto u_1, \ldots, z_k \mapsto u_k\}$ is an mgu of $s \doteq t$.

LEMMA A.1. *Let $s[x]$ and $t[y]$ be two non-ground non-trivial one-variable terms, and $x \neq y$. Let U be the set of non-ground strict subterms of s and t and let V be the set of ground strict subterms of s and t. If $s[x]$ and $t[y]$ are unifiable then they have a mgu σ such that one of the following is true:*

—$\sigma = \{x \mapsto u[y]\}$ *where* $u \in U$.
—$\sigma = \{y \mapsto u[x]\}$ *where* $u \in U$.
—$\sigma = \{x \mapsto u, y \mapsto v\}$ *where* $u, v \in U[V]$.

PROOF. Note that $V \subseteq U[V]$ since U contains the trivial terms also. We use the above unification algorithm. We start with the multiset $M_0 = s \doteq t$. We claim that if $M_0 \to^+ M$ then M is of one of the following forms:

(1) $s_1[x] \doteq t_1[y], \ldots, s_n[x] \doteq t_n[y]$, where each $s_i, t_i \in U \cup V$, some $s_i \in U$ and some $t_j \in U$.
(2) $s_1[u_1[y']] \doteq t_1[y'], \ldots, s_n[u_n[y']] \doteq t_n[y'], x' \doteq u[y']$ where $u \in U$, each u_i is a subterm of u, each $s_i, t_i \in U \cup V$, $x' \in \{x, y\}$ and $y' \in \{x, y\} \setminus \{x'\}$.
(3) $s_1[u_1] \doteq t_1[y'], \ldots, s_n[u_n] \doteq t_n[y'], x' \doteq u$ where $u \in V$, each u_i is a subterm of u, each $s_i, t_i \in U \cup V$, some $t_i \in U$, $x' \in \{x, y\}$ and $y' \in \{x, y\} \setminus \{x'\}$.
(4) $M', x \doteq u, y \doteq v$ where $u, v \in U[V]$, and no variables occur in M'.
(5) $\bot$.

As s and t are non-trivial, and x and y are distinct, hence (Delete) and (Bind) do not apply on M_0. Applying (Decomp) on M_0 leads us to type (1). Applying (Fail1) or (Fail2) on any M leads us to $\bot$. Applying (Delete) and (Decomp) on type (1) keeps us in type (1). Applying (Bind) on type (1) leads to type (2) or (3) depending on whether the concerned variable is replaced by a non-ground or ground term. Applying (Delete) on type (2) leads to type (2) itself. Applying (Decomp) on type (2) leads to type (2) itself. (Bind) applies on M of type (2) only if M contains some $y' \doteq v$ where v is ground. We must have $v \in V$. The result is of type (4). Applying (Delete) and (Decomp) rules on type (3) leads to type (3) itself. (Bind) applies on M of type (3) only if M contains some $y' \doteq v$ where v is ground. We must have $v \in U[V]$. The result is of type (4). Applying (Delete) and (Decomp) on type (4) leads to type (4) itself, and (Bind) does not apply.

Now we look at the solved forms. Solved forms of type (1) are of the form either $x \doteq u[y]$ with $u \in U$, or $y \doteq u[x]$ with $u \in U$, or $x \doteq u, y \doteq v$ with $u, v \in V \subseteq U[V]$. M of type (2) is in solved form only if $n = 0$. Hence the solved forms are again of the form $x \doteq u[y]$ or $y \doteq u[x]$ with $u \in U$. M of type (3) is in solved form only if $n = 1$, hence M

is of the form $x \doteq u, y \doteq v$ with $u, v \in U[V]$. Solved forms of type (4) are again of type $x \doteq u, y \doteq v$ with $u, v \in U[V]$ (i.e. M' is empty). □

PROOF OF LEMMA 5.1. By Lemma A.1, $s[x]$ and $t[y]$ have a mgu σ' such that one of the following is true:

—$\sigma' = \{x \mapsto u[y]\}$ where $u \in U$. We have $s[u[y]] = t[y]$. As t is irreducible and s is non-trivial, this is possible only if u is trivial. Hence $s[y] = t[y]$, so $s[x] = t[x]$. This is a contradiction.

—$\sigma' = \{y \mapsto u[x]\}$ where $u \in U$. This case is similar to the previous case.

—$\sigma' = \{x \mapsto u, y \mapsto v\}$ where $u, v \in U[V]$. As σ' is the mgu and maps x and y to ground terms, hence $\sigma = \sigma'$.

□

PROOF OF LEMMA 5.2. We use the above unification algorithm. We start with the multiset $M_0 = s[x] \doteq t[x]$. If $M_0 \rightarrow^+ M$ then M is of one of the following forms:

(1) $s_1[x] \doteq t_1[x], \ldots, s_n[x] \doteq t_n[x]$ where each s_i is a strict subterm of s and each t_i is a strict subterm of t

(2) $M, x \doteq u$ where u is a ground strict subterm of s or t, and no variables occur in M

(3) $\bot$.

Then it is easy to see that the only possible solved forms are the empty multiset, or $x \doteq u$ where u is a ground strict subterm of s or t. But the empty multiset cannot give us a unifier since $s[x] \neq t[x]$. □

B. PROOFS OF SECTION 7

PROOF OF THEOREM 7.1. A *standard Herbrand interpretation* is a Herbrand interpretation $\mathcal{H}$ such that $\overline{C} \in \mathcal{H}$ iff $\mathcal{H}$ does not satisfy C. This leads us to the notion of *standard satisfiability* as expected. The given set S of $\mathbb{P}$-clauses is satisfiable iff it is standard-satisfiable. Ordered resolution, factorization and splitting preserve satisfiability in any given Herbrand interpretation, and $\mathcal{Q}$-splitting preserves satisfiability in any given standard-Herbrand interpretation. Also if $T \rightarrow_{\mathcal{R}} T'$ then $T \cup cl(\mathcal{R})$ is satisfiable in a Herbrand interpretation iff $T' \cup cl(\mathcal{R})$ is satisfiable in that interpretation. This proves correctness: if $S \Rrightarrow^*_{<_s,\phi,\mathcal{R}} \mathcal{T}$ and $\mathcal{T}$ is closed then $S \cup cl(\mathcal{R})$ is unsatisfiable.

For completeness we replay the proof of [Goubault-Larrecq 2004] for ordered resolution with selection specialized to our case, and insert the arguments required for the replacement rules. Since $<$ is enumerable, we have an enumeration $A'_1, A'_2, \ldots$ of all ground atoms such that if $A'_i < A'_j$ then $i < j$. Also there are only finitely many splitting atoms in $\mathcal{Q}$, all of which are smaller than non-splitting atoms. Hence the set of all (splitting as well as non-splitting) atoms can be enumerated as $A_1, A_2, \ldots$ such that if $A_i <_s A_j$ then $i < j$. Clearly all the splitting atoms occur before the non-splitting atoms in this enumeration. Consider the infinite binary tree $\mathbb{T}$ whose nodes are literal sequences of the form $\pm_1 A_1 \pm_2 A_2 \ldots \pm_k A_k$ for $k \geq 0$. The two successors of the node N are $N + A_{k+1}$ (the left child) and $N - A_{k+1}$ (the right child). If $k = 0$ then N is a root node. Furthermore we write $-N = \mp_1 A_i \mp_2 A_2 \ldots \mp_k A_k$. These trees are known as semantic trees in the literature [Joyner Jr. 1976]. A clause C *fails* at a node N if there is some ground substitution σ such that for every literal $L \in C$, $L\sigma$ is in $-N$. For any set T of clauses

define $\mathbb{T}_T$ as the tree obtained from $\mathbb{T}$ by deleting the subtrees below all nodes of $\mathbb{T}$ where some clause of T fails. A failure-witness for a set T of clauses is a tuple $(\mathbb{T}', C_\bullet, \theta_\bullet)$ such that $\mathbb{T}' = \mathbb{T}_T$ is finite, C_N is a clause of T for each leaf node N of $\mathbb{T}'$, and θ_N is a ground substitution for each leaf node N of $\mathbb{T}'$ such that for $-N$ contains every $L \in C_N\theta_N$. We define $\nu(\mathbb{T}')$ as the number of nodes in $\mathbb{T}'$. For any failure witness of the form $(\mathbb{T}', C_\bullet, \theta_\bullet)$ and for any leaf node $N = \pm_1 A_1 \pm_2 A_2 \ldots \pm_k A_k$ of $\mathbb{T}'$, define $\mu_1(C_N, \theta_N)$ as follows:
– If $C_N \notin cl(\mathcal{R})$ then $\mu_1(C_N, \theta_N)$ is the multiset of integers which contains the integer i as many times as there are literals $\pm A' \in C_N$ such that $A'\theta_N = A_i$.
– If $C_N \in cl(\mathcal{R})$ then $\mu_1(C_N, \theta_N)$ is the empty multiset.
We define $\mu^-(\mathbb{T}', C_\bullet, \theta_\bullet)$ as the multiset of the values $\mu_1(C_N, \theta_N)$ where N ranges over all leaf nodes of $\mathbb{T}'$. We define $\mu(\mathbb{T}', C_\bullet, \theta_\bullet) = (\nu(\mathbb{T}'), \mu^-(\mathbb{T}', C_\bullet, \theta_\bullet))$. We consider the lexicographic ordering on pairs, i.e. $(x_1, y_1) < (x_2, y_2)$ iff either $x_1 < x_2$, or $x_1 = x_2$ and $y_1 < y_2$. Since $S \cup cl(\mathcal{R})$ is unsatisfiable, from K¨onig's Lemma [Kleene 2002]:

LEMMA B.1. *$S \cup cl(\mathcal{R})$ has a failure witness.*

LEMMA B.2. *If T has a failure witness $(\mathbb{T}_T, C_\bullet, \theta_\bullet)$ such that $\mathbb{T}_T$ is not just the root node, then there is some T' with a failure witness $(\mathbb{T}_{T'}, C'_\bullet, \theta'_\bullet)$ such that $T \Rrightarrow_{<_s} T'$ and $\mu(\mathbb{T}_{T'}, C'_\bullet, \theta'_\bullet) < \mu(\mathbb{T}_T, C_\bullet, \theta_\bullet)$.*

PROOF. We generalize the notion of mgu as usual, and we write $mgu(s_1 \doteq \ldots \doteq s_n)$ for the most general substitution which makes $s_1, \ldots, s_n$ equal. We iteratively define a sequence $R_0, R_1, \ldots$ of nodes, none of which is a leaf node. R_0 is the empty sequence which is not a leaf node. Suppose we have already defined R_i. As R_i is not a leaf node, R_i has a descendant N_i such that $N_i - B_i$ is rightmost leaf node in the subtree of $\mathbb{T}_T$ rooted at R_i.

(1) If B_i is a non-splitting atom then stop the iteration.
(2) Otherwise B_i is a splitting atom.
  (2a) If the subtree rooted at $N_i + B_i$ has some leaf node N such that $-B_i \in C_N$ then stop the iteration.
  (2b) Otherwise $N_i + B_i$ cannot be a leaf node, by definition of failure witnesses. Define $R_{i+1} = N_i + B_i$ and continue the iteration.

$\mathbb{T}_T$ is finite hence the iteration terminates. Let k be the largest integer for which R_k, and hence N_k and B_k are defined. For $0 \leq i \leq k-1$, B_i is a splitting literal. The only positive literals in the sequence N_k are from the set $\{B_0, \ldots, B_{k-1}\}$, since we followa left branch only in step (2b). $N_k - B_k$ is a leaf node of $\mathbb{T}_T$.

Suppose the iteration stopped in case (1) above. Then N_k has some descendant N such that its two children $N - B$ and $N + B$ are leaf nodes of $\mathbb{T}_T$, and B is a non-splitting literal. As B_k is a non-splitting literal, no negative splitting literals are present in C_{N-B} or C_{N+B}. C_{N-B} is of the form $C_1 \vee B'_1 \vee \ldots \vee B'_m (m \geq 1)$ such that $B'_1\theta_{N-B} = \ldots = B'_m\theta_{N-B} = B$ and each literal in $C_1\theta_{N-B}$ is present in $-N$. The literals $B'_1, \ldots, B'_m$ are then maximal in C_{N-B} and can be selected for resolution. C_{N+B} is of the form $C_2 \vee -B''_1 \vee \ldots \vee -B''_n (n \geq 1)$ such that $B''_1\theta_{N+B} = \ldots = B''_n\theta_{N+B} = B$ and each literal in $C_2\theta_{N+B}$ is present in $-N$. The literals $B''_1, \ldots, B''_n$ are then maximal in C_{N+B} and can be selected for resolution. We assume that C_{N-B} and C_{N+B} are renamed apart so as not to share variables. Let θ be a ground substitution which maps each $x \in \mathsf{fv}(C_{N-B})$ to $x\theta_{N-B}$ and $x \in \mathsf{fv}(C_{N+B})$ to $x\theta_{N+B}$. We have $B'_1\theta = \ldots = B'_m\theta = B''_1\theta = \ldots = B''_n\theta$.

Then $\sigma = mgu(B'_1 \doteq \ldots \doteq B'_m \doteq B''_1 \doteq \ldots \doteq B''_n)$ exists. Hence we have some ground substitution θ' such that $\sigma\theta' = \theta$. Hence by repeated applications of the ordered factorization and ordered binary resolution rule, we obtain the resolvent $C = C_1\sigma \vee C_2\sigma$, and $T \Rrightarrow_{<_s} T' = T \cup \{C\}$. We have $C\theta' = C_1\theta \vee C_2\theta$. Hence C fails at node N. Then $\mathbb{T}_{T'}$ is finite and $\nu(\mathbb{T}_{T'}) < \nu(\mathbb{T}_T)$. Hence by choosing any $C'_\bullet$ and $\theta'_\bullet$ such that $(\mathbb{T}_{T'}, C'_\bullet, \theta'_\bullet)$ is a failure witness for T', we have $\mu(\mathbb{T}_{T'}, C'_\bullet, \theta'_\bullet) < \mu(\mathbb{T}_T, C_\bullet, \theta_\bullet)$.

If the iteration did not stop in case (1) but in case (2a) then it means that B_k is a splitting literal. Then $C_{N_k - B_k} = C_1 \vee +B_k$ (with $B_k \notin C_1$). C_1 has no negative splitting literals. Hence the only literals in C_1 are positive splitting literals. Hence the literal B_k can be chosen from $C_{N_k - B_k}$ for resolution. The subtree rooted at $N_k + B_k$ has some leaf node N such that $-B_k \in C_N$. Then $C_N = C_2 \vee -B_k$ (and $-B_k \notin C_2$). Hence $-B_k$ can be selected from C_N for resolution. We obtain the resolvent $C_2 \vee C_1$ which fails at N. Let $T' = T \cup \{C_1 \vee C_1\}$. We have $\nu(\mathbb{T}_{T'}) \leq \nu(\mathbb{T}_T)$. If N' is the highest ancestor of N where $C_2 \vee C_1$ fails then N' is a leaf of $\mathbb{T}_{T'}$ and we define $C'_{N'} = C_2 \vee C_1$ and $\theta'_{N'} = \theta_N$. We have $\mu_1(C'_{N'}, \theta'_{N'}) < \mu_1(C_N, \theta_N)$ since all literals in C_1 are splitting literals $\pm q$ such that q occurs strictly before B_k in the enumeration $A_1, A_2, \ldots$. (Also note that $C_N \notin cl(\mathcal{R})$ because C_N contains a splitting literal). All other leaf nodes N'' of $\mathbb{T}_{T'}$ are also leaf nodes of $\mathbb{T}_T$ and we define $C'_{N''} = C_{N''}$ and $\theta'_{N''} = \theta_{N''}$. Then $(\mathbb{T}_{T'}, C'_\bullet, \theta'_\bullet)$ is a failure witness for T' and we have $\mu^-(\mathbb{T}_{T'}, C'_\bullet, \theta'_\bullet) < \mu^-(\mathbb{T}_T, C_\bullet, \theta_\bullet)$. Hence we have $\mu(\mathbb{T}_{T'}, C'_\bullet, \theta'_\bullet) < \mu(\mathbb{T}_T, C_\bullet, \theta_\bullet)$. □

LEMMA B.3. *If T has a failure witness $(\mathbb{T}_T, C_\bullet, \theta_\bullet)$ and $T \rightarrow_{\mathcal{Q}-nspl} T'$ then $T' \cup cl(\mathcal{R})$ has a failure witness $(\mathbb{T}_{T' \cup cl(\mathcal{R})}, C'_\bullet, \theta'_\bullet)$ with $\mu(\mathbb{T}_{T' \cup cl(\mathcal{R})}, C'_\bullet, \theta'_\bullet) \leq \mu(\mathbb{T}_T, C_\bullet, \theta_\bullet)$.*

PROOF. Let $C = C_1 \sqcup C_2 \in T$, C_2 is a non-empty $\mathbb{P}$-clause, C_1 has at least one non-splitting literal, and $T \rightarrow_{\mathcal{Q}-nspl} T' = (T \setminus \{C\}) \cup \{C_1 \vee -\overline{C_2}, \overline{C_2} \vee C_2\}$. If $C \neq C_N$ for any leaf node N of $\mathbb{T}_T$ then there is nothing to show. Now suppose $C = C_N$ where N is a leaf node of $\mathbb{T}_T$. If $C_N \in cl(\mathcal{R})$ then there is nothing to prove. Now suppose $C_N \notin cl(\mathcal{R})$. As C is constrained to contain at least one non-splitting literal, hence the literal sequence N has at least one non-splitting literal. By the chosen enumeration $A_1, A_2, \ldots$, either $\overline{C_2}$ or $-\overline{C_2}$ occurs in the literal sequence N.

—If $\overline{C_2}$ occurs in N then $C_1 \vee -\overline{C_2}$ fails at N. Let N' be the highest ancestor of N where it fails. N' is a leaf node of $\mathbb{T}_{T'}$. We define $C''_{N'} = C_1 \vee -\overline{C_2}$ and $\theta''_{N'} = \theta_N$. All other leaf nodes N'' of $\mathbb{T}_{T'}$ are also leaf nodes of $\mathbb{T}_T$ and we define $C''_{N''} = C_{N''}$ and $\theta''_{N''} = \theta_{N''}$. $(\mathbb{T}_{T'}, C''_\bullet, \theta''_\bullet)$ is a failure witness for T'. As C_2 has at least one non-splitting literal, we have $\mu_1(C''_{N'}, \theta''_{N'}) < \mu_1(C_N, \theta_N)$ (recall that $C_N \notin cl(\mathcal{R})$) so that $\mu(\mathbb{T}_{T'}, C''_\bullet, \theta''_\bullet) \leq \mu(\mathbb{T}_T, C_\bullet, \theta_\bullet)$. As $T' \subseteq T' \cup cl(\mathcal{R})$ hence the result follows.

—If $-\overline{C_2}$ occurs in N then $C_2 \vee \overline{C_2}$ fails at N. Since C_1 has at least one non-splitting literal, as in the previous case, we obtain a failure witness $(\mathbb{T}_{T'}, C''_\bullet, \theta''_\bullet)$ such that $\mu(\mathbb{T}_{T'}, C''_\bullet, \theta''_\bullet) \leq \mu(\mathbb{T}_T, C_\bullet, \theta_\bullet)$.

□

LEMMA B.4. *If T has a failure witness $(\mathbb{T}_T, C_\bullet, \theta_\bullet)$ and $T \rightarrow_{spl} T_1 \mid T_2$ then $T_1 \cup cl(\mathcal{R})$ and $T_2 \cup cl(\mathcal{R})$ have failure witnesses $(\mathbb{T}_{T_1 \cup cl(\mathcal{R})}, C'_\bullet, \theta'_\bullet)$ and $(\mathbb{T}_{T_2 \cup cl(\mathcal{R})}, C''_\bullet, \theta''_\bullet)$ such that $\mu(\mathbb{T}_{T_1 \cup cl(\mathcal{R})}, C'_\bullet, \theta'_\bullet) \leq \mu(\mathbb{T}_T, C_\bullet, \theta_\bullet)$ and $\mu(\mathbb{T}_{T_2 \cup cl(\mathcal{R})}, C''_\bullet, \theta''_\bullet) \leq \mu(\mathbb{T}_T, C_\bullet, \theta_\bullet)$.*

PROOF. Let $C = C_1 \sqcup C_2 \in T$ such that C_1 and C_2 share no variables, and we have $T \rightarrow_{spl} T_1 \mid T_2$ where $T_i = T \cup \{C_i\}$. We prove the required result for T_1, the other part is symmetric. If $C \neq C_N$ for any leaf node N of $\mathbb{T}_T$ then there is nothing to show. Now suppose $C = C_N$ for some leaf node N of $\mathbb{T}_T$. If $C_N \in cl(\mathcal{R})$ then there is nothing to show. Now suppose $C_N \notin cl(\mathcal{R})$. Since $C_1 \subseteq C$, hence C_1 also fails at N. Let N' be the highest ancestor of N where C_1 fails. N' is a leaf node of $\mathbb{T}_{T_1}$. We define $C'''_{N'} = C$ and $\theta'''_{N'} = \theta$. All other leaf nodes N'' of $\mathbb{T}_{T_1}$ are also leaf nodes of $\mathbb{T}_T$, and we define $C'''_{N''} = C_{N''}$ and $\theta'''_{N''} = \theta_{N''}$. $(\mathbb{T}_{T_1}, C'''_\bullet, \theta'''_\bullet)$ is a failure witness for T_1. Also $\mu_1(C'''_{N'}, \theta'''_{N'}) \leq \mu_1(C_N, \theta_N)$ (recall that $C_N \notin cl(\mathcal{R})$). Hence $\mu(\mathbb{T}_{T'}, C'''_\bullet, \theta'''_\bullet) \leq \mu(\mathbb{T}_T, C_\bullet, \theta_\bullet)$. As $T_1 \subseteq T_1 \cup cl(\mathcal{R})$, hence the result follows. □

The following arguments are the ones that take care of replacement steps.

LEMMA B.5. *If T has a failure witness $(\mathbb{T}_T, C_\bullet, \theta_\bullet)$ and $T \rightarrow_{\mathcal{R}} T'$ then $T' \cup cl(\mathcal{R})$ has a failure witness $(\mathbb{T}_{T' \cup cl(\mathcal{R})}, C'_\bullet, \theta'_\bullet)$ with $\mu(\mathbb{T}_{T' \cup cl(\mathcal{R})}, C'_\bullet, \theta'_\bullet) \leq \mu(\mathbb{T}_T, C_\bullet, \theta_\bullet)$.*

PROOF. Let $C_1 = C'_1 \vee \pm A\sigma \in T$, $R = A \rightarrow B \in \mathcal{R}$, and $T \rightarrow_{\mathcal{R}} T' = (T \setminus \{C_1\}) \cup \{C\}$ where $C = C'_1 \vee \pm B\sigma$. If $C_1 \neq C_N$ for any leaf node of $\mathbb{T}_T$ then there is nothing to prove. Now suppose that $C_1 = C_N$ for some leaf node N of $\mathbb{T}_T$. Let $N = \pm_1 A_1 \ldots \pm_k A_k$. If $C_1 \in cl(\mathcal{R})$ then $T \subseteq T' \cup cl(\mathcal{R})$, and there is nothing to prove. Now suppose $C_1 \notin cl(\mathcal{R})$. We have a ground substitution θ such that $C_1\theta = C'_1\theta \vee \pm A\sigma\theta \subseteq \{\mp_1 A_1, \ldots, \mp_k A_k\}$. As $\mathcal{R}$ is ordered we have $A \geq B$. Hence $A\sigma\theta \geq B\sigma\theta$. Hence either $\pm B\sigma\theta \in \{\mp_1 A_1, \ldots, \mp_k A_k\}$ or $\mp B\sigma\theta \in \{\mp_1 A_1, \ldots, \mp_k A_k\}$.

—Suppose $\pm B\sigma\theta \in \{\mp_1 A_1, \ldots, \mp_k A_k\}$. Since $C_1\theta = C'_1\theta \vee \pm A\sigma\theta \subseteq \{\mp_1 A_1, \ldots, \mp_k A_k\}$, hence $C\theta = C'_1\theta \vee \pm B\sigma\theta \subseteq \{\mp_1 A_1, \ldots, \mp_k A_k\}$. Hence C fails at N. Let N' be the highest ancestor of N where C fails. N' is a leaf node of $\mathbb{T}_{T'}$. We define $C''_{N'} = C$ and $\theta''_{N'} = \theta$. All other leaf nodes N'' of $\mathbb{T}_{T'}$ are also leaf nodes of $\mathbb{T}_T$, and we define $C''_{N''} = C_{N''}$ and $\theta''_{N''} = \theta_{N''}$. $(\mathbb{T}_{T'}, C''_\bullet, \theta''_\bullet)$ is a failure witness for T'. Also $\mu_1(C''_{N'}, \theta''_{N'}) \leq \mu_1(C_N, \theta_N)$ (recall that $C_N \notin cl(\mathcal{R})$). Hence $\mu(\mathbb{T}_{T'}, C'_\bullet, \theta'_\bullet) \leq \mu(\mathbb{T}_T, C_\bullet, \theta_\bullet)$. As $T' \subseteq T' \cup cl(\mathcal{R})$, hence the result follows.

—Suppose $\mp B\sigma\theta \in \{\mp_1 A_1, \ldots, \mp_k A_k\}$. Since $\pm A\sigma\theta = \in \{\mp_1 A_1, \ldots, \mp_k A_k\}$, hence the clause $\mp A \vee \pm B \in cl(\mathcal{R})$ fails at N. Let N' be the highest ancestor of N where $\mp A \vee \pm B$ fails. N' is a leaf node of $\mathbb{T}_{T' \cup \{\mp A \vee \pm B\}}$. We define $C''_{N'} = C$ and $\theta''_{N'} = \theta$. All other leaf nodes N'' of $\mathbb{T}_{T' \cup \{\mp A \vee \pm B\}}$ are also leaf nodes of $\mathbb{T}_T$, and we define $C''_{N''} = C_{N''}$ and $\theta''_{N''} = \theta_{N''}$. $(\mathbb{T}_{T' \cup \{\mp A \vee \pm B\}}, C''_\bullet, \theta''_\bullet)$ is a failure witness for $T' \cup \{\mp A \vee \pm B\}$. Also $\mu_1(C''_{N'}, \theta''_{N'}) \leq \mu_1(C_N, \theta_N)$ since $\mu_1(C''_{N'}, \theta''_{N'})$ is the empty multiset. Hence $\mu(\mathbb{T}_{T' \cup \{\mp A \vee \pm B\}}, C''_\bullet, \theta''_\bullet) \leq \mu(\mathbb{T}_T, C_\bullet, \theta_\bullet)$. As $T' \cup \{\mp A \vee \pm B\} \subseteq T' \cup cl(\mathcal{R})$, hence the result follows.

□

For a tableaux $\mathcal{T} = S_1 \mid \ldots \mid S_n$, define $\mathcal{T} \cup S = S_1 \cup S \mid \ldots \mid S_n \cup S$. We define a failure witness for such a $\mathcal{T}$ to be a multiset $\{(\mathbb{T}_{S_1}, C^1_\bullet, \theta^1_\bullet), \ldots, (\mathbb{T}_{S_n}, C^1_\bullet, \theta^n_\bullet)\}$ where each $(\mathbb{T}_{S_i}, C^i_\bullet, \theta^i_\bullet)$ is a failure witness of S_i. We define

$$\mu(\{\mathbb{T}_{S_1}, C^1_\bullet, \theta^1_\bullet), \ldots, (\mathbb{T}_{S_n}, C^1_\bullet, \theta^n_\bullet\}) = \{\mu(\mathbb{T}_{S_1}, C^1_\bullet, \theta^1_\bullet), \ldots, \mu(\mathbb{T}_{S_n}, C^1_\bullet, \theta^n_\bullet)\}.$$

Then it is clear that $S \cup cl(\mathcal{R})$ has a failure witness and whenever any $\mathcal{T}$ has a failure witness in which one of the trees has at least two nodes, then $\mathcal{T} \Rrightarrow_{<_s, \phi, \mathcal{R}} \mathcal{T}'$ for some $\mathcal{T}'$ such that $\mathcal{T}' \cup cl(\mathcal{R})$ has a strictly smaller failure witness. Hence we have some $\mathcal{T}$ such

that $S \Rrightarrow^*_{<_s,\phi,\mathcal{R}} \mathcal{T}$ and $\mathcal{T} \cup cl(\mathcal{R})$ has a failure witness in which each tree is a root node. Then $\mathcal{T} \cup cl(\mathcal{R})$ is closed. Hence $\mathcal{T}$ is closed.

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